Lecture 4
$O(3)$-model: mass generation by instantons

Consider the $O(3)$-model:

$$S[\boldsymbol{n}] = \frac{1}{2g}\int d^2x\,(\partial_\mu \boldsymbol{n})^2, \qquad n_1^2 + n_2^2 + n_3^2 = 1. \tag{1}$$

We will be interested in the functions $\boldsymbol{n}(x)$ with finite action. These function correspond to processes of finite amplitude $e^{-S[\boldsymbol{n}]}$. It follows from the finiteness of the action that the solution should tend to a constant at infinity:

$$\boldsymbol{n}(x) \underset{x\to\infty}{\to} \boldsymbol{n}_0. \tag{2}$$

Thus, the infinity can be considered as a point on the two-dimensional sphere S^2. The continuous function $\boldsymbol{n}(x)$, which satisfies the condition (2), defines a continuous mapping

$$\boldsymbol{n} : S^2 \to S^2. \tag{3}$$

Such mappings are classified by the *topological number* q, such that $\boldsymbol{n}$ realizes a $|q|$-sheeted covering of a sphere by a sphere. If $q > 0$, the mapping preserves orientation, while if $q < 0$, reverses it. A representative of the class q is, for example, a mapping:

$$\theta' = \theta, \qquad \varphi' = q\varphi, \tag{4}$$

where (θ, φ) are the polar coordinates on a sphere.

Let us express the topological number in terms of the field $\boldsymbol{n}$. We will consider it as a mapping of the sphere S^2 onto the sphere $S^{2\prime}$. On the sphere S^2 we set the coordinates $x = (x^1, x^2)$, while on the sphere $S^{2\prime}$ the coordinates $x' = (x'^1, x'^2)$ and the metric $g'_{\mu\nu}$. The metric on the sphere S^2 is not essential. Let

$$S = \int_{S^{2\prime}} d^2x' \sqrt{g'}$$

is the area of the sphere $S^{2\prime}$. Then the topological number can be expressed in terms of the area covered by the map $x'(x)$ when x covers the whole sphere S^2:

$$q = \frac{1}{S}\int_{x'(S^2)} d^2x'\,\sqrt{g'} = \frac{1}{S}\int_{S^2} d^2x \frac{\partial(x')}{\partial(x)}\sqrt{g'}.$$

In spherical coordinates

$$q = \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, \frac{\partial(\theta',\varphi')}{\partial(\theta,\varphi)}\sin\theta' = \frac{1}{4\pi}\int d^2x\, \frac{\partial(\theta',\varphi')}{\partial(x^1,x^2)}\sin\theta'.$$

To find the Jacobian $\partial(\theta',\varphi')/\partial(x^1,x^2) = (\partial\theta'/\partial x^1)(\partial\varphi'/\partial x^2) - (\partial\theta'/\partial x^2)(\partial\varphi'/\partial x^1)$, rewrite the spherical variables in terms $\boldsymbol{n}$ and x. Put

$$\boldsymbol{n} = (\sin\theta'\cos\varphi', \sin\theta'\sin\varphi', \cos\theta'). \tag{5}$$

It can be checked by a direct calculation that

$$\frac{1}{2}\boldsymbol{n}\,(\partial_\mu \boldsymbol{n} \times \partial_\nu \boldsymbol{n})\epsilon^{\mu\nu} = \frac{\partial(\theta',\varphi')}{\partial(x^1,x^2)}\sin\theta'. \tag{6}$$

From this we obtain

$$q = \frac{1}{8\pi}\int d^2x\, \boldsymbol{n}\,(\partial_\mu \boldsymbol{n} \times \partial_\nu \boldsymbol{n})\epsilon^{\mu\nu}. \tag{7}$$

It is possible to derive (7) even easier. The vector $\partial_1 \boldsymbol{n}\, dx^1$ is a small displacement vector on a sphere that corresponds to the displacement by $(dx^1, 0)$ in the x-space. Similarly, $\partial_2 \boldsymbol{n}\, dx^2$ is a displacement vector on a sphere that corresponds to the displacement by $(0, dx^2)$.Both vectors are perpendicular to the vector $\boldsymbol{n}$.

Then $|\partial_1 \boldsymbol{n} \times \partial_2 \boldsymbol{n}\, dx^1\, dx^2| = \pm \boldsymbol{n}\, (\partial_1 \boldsymbol{n} \times \partial_2 \boldsymbol{n})\, d^2x$ is the area of a small parallelogram on the sphere. The "+" sign corresponds to the orientation preserving map, and the "−" sign corresponds to the orientation reversing map. Therefore,

$$\int d^2x\, \boldsymbol{n}\, (\partial_1 \boldsymbol{n} \times \partial_2 \boldsymbol{n}) = \frac{1}{2} \int d^2x\, \boldsymbol{n}\, (\partial_\mu \boldsymbol{n} \times \partial_\nu \boldsymbol{n}) \epsilon^{\mu\nu}$$

is equal to the area on the sphere, covered during the integration over the entire x-space, which is equal to $4\pi q$.

From the identity

$$\int d^2x\, (\partial_\mu \boldsymbol{n} + \epsilon_{\mu\nu} \boldsymbol{n} \times \partial^\nu \boldsymbol{n})^2 = 2 \int d^2x\, (\partial_\mu \boldsymbol{n})^2 - 2 \int d^2x\, \boldsymbol{n}\, (\partial_\mu \boldsymbol{n} \times \partial_\nu \boldsymbol{n}) \epsilon^{\mu\nu} \tag{8}$$

we obtain

$$S[\boldsymbol{n}] = \frac{4\pi q}{g} + \frac{1}{4g} \int d^2x\, (\partial_\mu \boldsymbol{n} + \epsilon_{\mu\nu} \boldsymbol{n} \times \partial^\nu \boldsymbol{n})^2. \tag{9}$$

By the substitution $\boldsymbol{n} \to -\boldsymbol{n}$, $q \to -q$ we have

$$S[\boldsymbol{n}] = -\frac{4\pi q}{g} + \frac{1}{4g} \int d^2x\, (\partial_\mu \boldsymbol{n} - \epsilon_{\mu\nu} \boldsymbol{n} \times \partial^\nu \boldsymbol{n})^2. \tag{10}$$

It obviously follows from this that

$$S[\boldsymbol{n}] \geq \frac{4\pi |q|}{g}. \tag{11}$$

Inequality (11) turns into an equality on solutions of the first-order equations (the *self-duality equations*)

$$\partial_\mu \boldsymbol{n} = -\epsilon_{\mu\nu} \boldsymbol{n} \times \partial^\nu \boldsymbol{n} \qquad (q \geq 0), \tag{12}$$

$$\partial_\mu \boldsymbol{n} = \epsilon_{\mu\nu} \boldsymbol{n} \times \partial^\nu \boldsymbol{n} \qquad (q \leq 0). \tag{13}$$

The solutions of these equations minimize the action and, therefore, constitute a subclass of the solutions to the equations of motion. To solve the self-duality equations explicitly, we use the stereographic projection onto the plane. It is convenient to use the complex coordinate w on this plane:

$$n_1 + i n_2 = \frac{2w}{1 + |w|^2}, \qquad n_3 = \frac{1 - |w|^2}{1 + |w|^2}. \tag{14}$$

Substituting (14) into (12), (13), we obtain

$$\bar{\partial} w = 0 \qquad (q \geq 0), \tag{15}$$

$$\partial w = 0 \qquad (q \leq 0). \tag{16}$$

Consider the case $q > 0$. The function $w(z)$ must be a meromorphic function on the sphere, that is, the only singularities of this function, including a singularity at infinity, are poles. The number of nodes and poles of such a function should be finite. The point $\boldsymbol{n} = (0, 0, 1)$ corresponds to nodes, while the point $\boldsymbol{n} = (0, 0, -1)$ to poles. The most general such solution (15) has the form

$$w(q, \vec{a}, \vec{b}, c; z) = c \prod_{j=1}^{q} \frac{z - a_j}{z - b_j}, \tag{17}$$

where $c \in \mathbb{C} \setminus \{0\}$, $a_j, b_j \in \mathbb{C} \cup \{\infty\}$, $a_i \neq b_j$ $(\forall i, j)$. Besides, the limiting cases $a_i \to \infty$, $c a_i = \text{const}$ and $b_i \to \infty$, $c/b_i = \text{const}$ should be taken into account, but these cases have zero measure. It is easy to find a topological number corresponding to the solution (17). Let $w_0 \in \mathbb{C} \cup \{\infty\}$. Then the number of solutions to the equation $w(z) = w_0$ accounting for multiplicities is independent of w_0. Let w_0 be equal, for example, to infinity. Then it is obvious that the number of solutions to this equation is equal to q. This means that $w(z)$ provides a q-sheeted covering and the topological number is q.

Similarly, in the case of a topological number $q < 0$ we obtain

$$w(q, \vec{a}, \vec{b}, c; \bar{z}) = c \prod_{j=1}^{-q} \frac{\bar{z} - a_j}{\bar{z} - b_j}, \tag{18}$$

The solutions (17) and (18) are called *(multi)instanton solutions.*

Let us try to use instanton solutions to approximately calculate the functional integral for the quantum $O(3)$-model. We will integrate over multi-instanton solutions and small fluctuations near them.

Rewrite (9) and (10) in terms of the field $w(z, \bar{z})$:

$$S[w, \bar{w}] = \frac{4\pi q}{g} + \frac{8}{g} \int d^2x \, \frac{\bar{\partial} w \partial \bar{w}}{(1 + |w|^2)^2} \tag{19}$$

$$= -\frac{4\pi q}{g} + \frac{8}{g} \int d^2x \, \frac{\partial w \bar{\partial} \bar{w}}{(1 + |w|^2)^2}. \tag{20}$$

Consider the case of $q \geq 0$. Let

$$S_q[\varphi, \bar{\varphi}] = S[w(q, \vec{a}, \vec{b}, c; z)(1 + g^{1/2}\varphi(z, \bar{z})), w^*(q, \vec{a}, \vec{b}, c; z)(1 + g^{1/2}\bar{\varphi}(z, \bar{z}))]. \tag{21}$$

It is easy to see that

$$S_q[\varphi, \bar{\varphi}] = \frac{4\pi q}{g} + 8 \int d^2x \, \frac{|w|^2}{(1 + |w|^2)^2} \bar{\partial}\varphi \partial \bar{\varphi} \tag{22}$$

in the quadratic approximation. Accordingly, the q-instanton contribution to the partition function has the form

$$Z_q = \frac{e^{-4\pi q/g}}{(q!)^2} \int d\mu(\vec{a}, \vec{b}, c) \, Z[w(q, \vec{a}, \vec{b}, c; z)], \qquad Z[w] = \int D\varphi \, D\bar{\varphi} \, \exp\left(-8 \int d^2x \, \frac{|w|^2}{(1 + |w|^2)^2} \bar{\partial}\varphi \partial \bar{\varphi}\right). \tag{23}$$

Here $d\mu(\vec{a}, \vec{b}, c)$ is the integration measure over $\vec{a}$, $\vec{b}$ and c, which is invariant under translations, extensions and inversions. This measure is easy to find:

$$d\mu(\vec{a}, \vec{b}, c) = k^q \frac{d^2c}{|c|^2} \prod_{j=1}^{q} d^2a_j \, d^2b_j \prod_{i<j} |a_i - a_j|^4 |b_i - b_j|^4 \prod_{i,j} |a_i - b_j|^{-4} \tag{24}$$

with a certain constant k.

Evidently the integral $Z[w]$ is independent of the coupling constant g. Due to the contribution of ultraviolet and infrared cutoffs, this integral will not be literally invariant with respect to the conformal transformations of the parameters a_j, b_j. It will be transformed by the rule:

$$Z[w] \to Z[w'] \prod_{j=1}^{q} \left|\frac{da'_j}{da_j}\right|^{2\alpha} \left|\frac{db'_j}{db_j}\right|^{2\alpha}$$

with a certain α. It follows that

$$Z[w] \sim f(c) \prod_{i<j} |a_i - a_j|^{-4\alpha} |b_i - b_j|^{-4\alpha} \prod_{i,j} |a_i - b_j|^{4\alpha}.$$

Certain rather complex calculations can show that $\alpha = 1/2$ and $f(c) = |c|^2/(1 + |c|^2)^2$, so the integral over c gives just a finite factor. If you accept this, you obtain [1]

$$Z_q \sim \frac{\lambda^q}{(q!)^2} \int \prod_{j=1}^{q} d^2a_j \, d^2b_j \prod_{i<j} |a_i - a_j|^2 |b_i - b_j|^2 \prod_{i,j} |a_i - b_j|^{-2}, \tag{25}$$

where $\lambda \sim e^{-4\pi/g}$.

It is easy to verify that this is a plasma described by the sine-Gordon model with $\beta^2 = 1$ or the Thirring model with $g = 0$, that is, a free massive fermion. This means that the instantons in the model generate a mass. Detailed calculations were performed in [2].

The fault of the above reasoning is that we only took into account the instanton solutions. We can, of course, take into account the anti-instanton solutions. However, a realistic description of the model can only be when using fluctuations near functions containing both instantons and anti-instantons. Unfortunately, such a description is not yet available.

There is one important consequence of the existence of a topological number (7). Indeed, the action of the $O(3)$-models can be modified:

$$S_\theta(\boldsymbol{n}) = S(\boldsymbol{n}) + i\theta q = \frac{1}{2g}\int d^2x\,(\partial_\mu \boldsymbol{n})^2 + i\frac{\theta}{8\pi}\int d^2x\,\boldsymbol{n}(\partial_\mu \boldsymbol{n} \times \partial_\nu \boldsymbol{n})\epsilon^{\mu\nu}. \tag{26}$$

The last term is called the *θ-term*. The physical properties of the model substantially depend on θ, since the partition function (and the generating functional) are written as

$$Z = \sum_{q=-\infty}^{\infty} e^{i\theta q} Z_q.$$

In particular, it is known that for $\theta = \pi$ the $O(3)$-model is massless, but not scale-invariant.

Bibliography

[1] A. A. Belavin, A. M. Polyakov, *Metastable states of two-dimensional isotropic ferromagnets*, JETP Letters 22 (1975) 245–247

[2] V. A. Fateev, I. V. Frolov, A. S. Shvarts, *Quantum Fluctuations of Instantons in the Nonlinear Sigma Model*, Nucl. Phys. B154 (1979) 1–20

[Both papers are available in Russian in the collection "Instantons, Strings and Conformal Field Theory", ed. A. A. Belavin, 2002].

Problems

1. By using the substitution (5) obtain (6).

2. Derive the identity (8).

3. Prove the equivalence of the equations (15), (16) to the equations (12), (13).

4. Express the topological charge q in terms of the function $w(z, \bar{z})$ in the form of two-dimensional integral.

5*. Let $\boldsymbol{n}(x) = (n_1(x), \ldots, n_N(x))$ be *complex* scalar fields on the plain restricted by the condition

$$|\boldsymbol{n}(x)|^2 \equiv \sum_{i=1}^{N} |n_i(x)|^2 = 1,$$

and $A_\mu(x)$ is a real (co)vector field. Consider the action

$$S[\boldsymbol{n}, A] = \frac{1}{2g}\int d^2x\,\overline{D_\mu \boldsymbol{n}} \cdot D_\mu \boldsymbol{n}, \qquad D_\mu = \partial_\mu + A_\mu.$$

Show that the quantity

$$q = \frac{1}{2\pi}\int d^2x\,\epsilon_{\mu\nu}\,\partial_\mu A_\nu$$

is an integer topological charge, and in each topological sector we have

$$S \geq \frac{\pi}{g}|q|.$$

Find the self-duality equations.