Lecture 5
$O(N)$-model: $1/N$-expansion

Consider the general $O(N)$-model in the Minkowski space:

$$S[\boldsymbol{n}] = \frac{1}{2g}\int d^2x\,(\partial_\mu \boldsymbol{n})^2, \qquad \boldsymbol{n}^2 = 1. \tag{1}$$

It is convenient to introduce an auxiliary field $\omega(x)$ and write down the action in the form

$$S[\boldsymbol{n},\omega] = \frac{1}{2g}\int d^2x\,((\partial_\mu \boldsymbol{n})^2 - \omega(\boldsymbol{n}^2 - 1)), \tag{2}$$

where now the vector $\boldsymbol{n}$ runs through any values in $\mathbb{R}^N$. Consider the functional integral

$$Z[J] = \int D\omega\, D\boldsymbol{n}\, e^{iS[\boldsymbol{n},\omega] + ig^{-1/2}\int d^2x\,\boldsymbol{J}\boldsymbol{n}}. \tag{3}$$

The integral over $\boldsymbol{n}$ is Gaussian. Take it. Notice that

$$iS[\boldsymbol{n},\omega] + ig^{-1/2}\int d^2x\,\boldsymbol{J}\boldsymbol{n} = -\frac{1}{2}\left(\frac{n_i}{g^{1/2}}, K(\omega)\delta_{ij}\frac{n_j}{g^{1/2}}\right) + \left(iJ_i, \frac{n_i}{g^{1/2}}\right) + i\int d^2x\,\frac{\omega}{2g},$$

where

$$K(\omega) = i(\partial_\mu^2 + \omega).$$

From this we obtain

$$Z[J] = \int D\omega\,(\det(\partial_\mu^2 + \omega))^{-N/2}\exp\left(i\int d^2x\,\frac{\omega}{2g} - \frac{1}{2}\int d^2x\,d^2x'\,J_i(x)G(x,x'|\omega)J_i(x')\right),$$

where $G(x,x'|\omega)$ is the solution of the equation

$$i(\partial_\mu^2 + \omega(x))G(x,x'|\omega) = \delta(x - x'). \tag{4}$$

Otherwise, the generating functional can be rewritten in the form

$$Z[J] = \int D\omega\,\exp\left(iS_{\text{eff}}[\omega] - \frac{1}{2}\int d^2x\,d^2x'\,J_i(x)G(x,x'|\omega)J_i(x')\right), \tag{5}$$

$$S_{\text{eff}}[\omega] = i\frac{N}{2}\operatorname{tr}\log(\partial_\mu^2 + \omega) + \int d^2x\,\frac{\omega}{2g}. \tag{6}$$

Find the saddle point of this integral as $N \to \infty$. Suppose the saddle point corresponds to

$$\omega(x) = \text{ const } = \omega_0.$$

Then

$$\begin{aligned}
\operatorname{tr}\log(\partial_\mu^2 + \omega_0) &= V\int \frac{d^2k}{(2\pi)^2}\log(\omega_0 - k^2 - i0) \\
&= iV\int_E \frac{d^2k}{(2\pi)^2}\log(\omega_0 + k^2) \\
&= \frac{iV}{2\pi}\int_0^\Lambda dk\,k\log(\omega_0 + k^2) = \frac{iV}{4\pi}\int_{\omega_0}^{\omega_0+\Lambda^2} du\,\log u = \frac{iV}{4\pi}\left[u\log\frac{u}{e}\right]_{\omega_0}^{\omega_0+\Lambda^2} \\
&= \frac{iV}{4\pi}\left((\omega_0 + \Lambda^2)\log\frac{\omega_0 + \Lambda^2}{e} - \omega_0\log\frac{\omega_0}{e}\right) = \frac{iV}{4\pi}\left(\omega_0\log\frac{\Lambda^2}{\omega_0} + \Lambda^2\log\frac{\omega_0 + \Lambda^2}{e}\right).
\end{aligned} \tag{7}$$

where Λ is an ultraviolet cutoff parameter. Under the logarithm sign, we neglected ω_0 in the expression $\omega_0 + \Lambda^2$ in the first term. We find

$$0 = \frac{dS[\omega_0]}{d\omega_0} = V\left(-\frac{N}{8\pi}\log\frac{\Lambda^2}{\omega_0} + \frac{1}{2g}\right).$$

From this we obtain

$$\omega_0 = m^2 = \Lambda^2 \exp\left(-\frac{4\pi}{Ng}\right). \tag{8}$$

We see that in the limit $\Lambda \to \infty$ also g should be tended to zero, in such a way that the value $\omega_0 = m^2$ remains finite. For beta-functions at large N we find

$$\frac{dg}{d\log\Lambda} = \beta(g) = -\frac{N}{2\pi}g^2. \tag{9}$$

It is important that, the parameter m of the dimension of mass arises in the theory. We will see now that this is indeed a mass. In the theory a *dynamic mass generation* takes place. At no scales the correlation functions will decrease in a power-law manner, and the presence of a dimensional parameter will be noticeable in correlation functions at any scales.

Let us now develop the perturbation theory in the parameter $1/N$. Represent $\omega(x)$ in the form

$$\omega(x) = m^2 + (2/N)^{1/2}\rho(x). \tag{10}$$

and expand the effective action in powers of $N^{-1/2}\rho(x)$:

$$\begin{aligned}
S_{\text{eff}}[\omega] &= \text{const } + i\frac{N}{2}\operatorname{tr}\log\left(1+(2/N)^{1/2}\rho(\partial_\mu^2+m^2)^{-1}\right) + \frac{1}{(2N)^{1/2}g}\operatorname{tr}\rho \\
&= \text{const } + i\frac{N}{2}\operatorname{tr}\log(1+i(2/N)^{1/2}\rho G) + \frac{1}{(2N)^{1/2}g}\operatorname{tr}\rho \\
&= \text{const } + \left(\frac{1}{(2N)^{1/2}g}\operatorname{tr}\rho - \left(\frac{N}{2}\right)^{1/2}\operatorname{tr}\rho G\right) - i\frac{N}{2}\sum_{n=2}^{\infty}\frac{(-i)^n(2/N)^{n/2}}{n}\operatorname{tr}(\rho G)^n.
\end{aligned}$$

Here G is the operator with the kernel $G(x,x') = G(x,x'|m^2)$.

The parenthesis in the last expression is equal to zero under the assumption that $\omega = m^2$ is a minimum. Let us check this assumption. We have

$$\begin{aligned}
\operatorname{tr}\rho &= \int d^2x\,\rho(x), \\
\operatorname{tr}\rho G &= \int d^2x\,\rho(x)G(x,x) = G(0,0)\int d^2x\,\rho(x) = G(0,0)\operatorname{tr}\rho \\
&= V\int\frac{d^2k}{(2\pi)^2}\frac{i}{k^2-m^2+i0}\operatorname{tr}\rho = \frac{V}{4\pi}\log\frac{\Lambda^2}{m^2}\operatorname{tr}\rho = (gN)^{-1}\operatorname{tr}\rho.
\end{aligned}$$

We see that indeed the parenthesis vanishes. By studying the next contribution ($n = 2$) one can make sure that the point $\omega = m^2$ is a local minimum. There is no way to prove rigorously that this minimum is absolute.

Finally we have

$$S_{\text{eff}}[\omega] = \text{const } - i\frac{N}{2}\sum_{n=2}^{\infty}\frac{(-i)^n(2/N)^{n/2}}{n}\int d^{2n}x\,\rho(x_1)G(x_1,x_2)\ldots\rho(x_n)G(x_n,x_1). \tag{11}$$

The expansion starts from a quadratic term of the form

$$\frac{i}{2}\int d^2x_1\,d^2x_2\,\rho(x_1)G(x_1,x_2)\rho(x_2)G(x_2,x_1).$$

Therefore, the propagator $D(x_1,x_2)$ of the field $\rho(x)$ is the kernel of the operator inverse to that with the kernel

$$D^{-1}(x_1,x_2) = G(x_1,x_2)G(x_2,x_1).$$

Now it is clear why we needed the factor $(2/N)^{1/2}$ before ρ. It allowed us to get rid of the coefficient $2/N$ in the propagator $D(x_1,x_2)$.

Passing to the momentum representation, we obtain

$$D(k) = \quad \overset{k}{\text{- - - - - - - - - -}} \quad = -\left(\int \frac{d^2q}{(2\pi)^2}\,\frac{1}{(q^2-m^2+i0)((q+k)^2-m^2+i0)}\right)^{-1}. \tag{12}$$

In addition, the operator $G(x,x'|\omega)$, which in (5), should also be expanded in $\rho(x)$:

$$G[\omega] = \frac{1}{G^{-1}+i(2/N)^{1/2}\rho} = \sum_{n=0}^{\infty}(-i)^n\left(\tfrac{2}{N}\right)^{n/2}G(\rho G)^n,$$

$$G(x_1,x_2|\omega) = \sum_{n=0}^{\infty}(-i)^n\left(\frac{2}{N}\right)^{n/2}\int d^{2n}y\, G(x_1,y_1)\rho(y_1)G(y_1,y_2)\dots\rho(y_n)G(y_n,x_2).$$

Represent $G(x_1,x_2)$ by a solid line:

$$G_{ij}(p) = \quad i \overset{p}{\text{―――――――}} j \quad = G(p)\delta_{ij} = \frac{i\delta_{ij}}{p^2-m^2+i0}. \tag{13}$$

If we also introduce the vertex

$$\underset{\text{―――――}}{i \;\;\vdots\;\; j} \quad = -i\left(\frac{2}{N}\right)^{1/2}\delta_{ij}, \tag{14}$$

the following rules of the diagram technique can be formulated:

1. A diagram consists of dashed lines (12), solid lines (13) and vertices (14).
2. The outer lines of a diagram can only be solid lines corresponding to the massive particles $\varphi_i = g^{-1/2}n_i$.
3. Closed loops of solid lines must contain at least three vertices.

We see that in this formulation the diagram technique does not contain the coupling constant g at all. The order of the diagram in $1/N$ is equal to $\frac{1}{2}V - L$, where V is the number of vertices, and L is the number of loops of solid lines. From the rule 3 it follows that the order of the diagram is always positive.

The relation between the coupling constant g, the mass m, and the cutoff parameter Λ can be refined using the relation

$$\left\langle \sum_{i=1}^{N}\varphi_i^2(x)\right\rangle = \frac{1}{g}.$$

For example, in the order $1/N$ one can obtain

$$m^2 = \Lambda^2\exp\left(-\frac{4\pi}{(N-2)g'}\right), \qquad \frac{1}{g'} = \frac{1}{g} + \frac{\Lambda^2}{4\pi m^2\log(\Lambda^2/m^2)}. \tag{15}$$

The correction to the inverse coupling constant is the contribution of the specific sigma-model quadratic divergence. The divergence becomes logarithmic, if we add to the action (2) a term of the form $\alpha\int d^2x\,\omega^2$, which "blurs" the delta-function in the functional integral.

Let us try now to calculate the S-matrix of the $O(N)$-model. Let us examine kinematics first. We have N particles of mass m. Let two such particles with momenta p_1 and p_2 be scattered on each other, forming two new particles of the same mass with momenta p_1' and p_2'. It is convenient to parameterize the momenta p_a by rapidities θ_a:

$$p_a = m\,\mathrm{sh}\,\theta_a, \qquad p_a' = m\,\mathrm{sh}\,\theta_a'.$$

Then

$$m\,\mathrm{ch}\,\theta_1 + m\,\mathrm{ch}\,\theta_2 = m\,\mathrm{ch}\,\theta_1' + m\,\mathrm{ch}\,\theta_2',$$
$$m\,\mathrm{sh}\,\theta_1 + m\,\mathrm{sh}\,\theta_2 = m\,\mathrm{sh}\,\theta_1' + m\,\mathrm{sh}\,\theta_2'.$$

These equations have just two solutions: $\theta_1' = \theta_1$, $\theta_2' = \theta_2$ and $\theta_1' = \theta_2$, $\theta_2' = \theta_1$. The scattering matrix of two particles into two can be presented as

$$S_{ij}^{i'j'}(\theta_1,\theta_2;\theta_1',\theta_2') = (2\pi)^2\delta(p_1'-p_1)\delta(p_2'-p_2)S_{ij}^{i'j'}(\theta_1-\theta_2) + (2\pi)^2\delta(p_2'-p_1)\delta(p_1'-p_2)S_{ij}^{j'i'}(\theta_1-\theta_2).$$

To properly normalize, one has to convert the delta-functions to the standard form of the delta-function over space-time momenta:

$$
\begin{aligned}
S_{ij}^{i'j'}(\theta_1, \theta_2; \theta_1', \theta_2') = (2\pi)^2 \delta^{(2)}(P' - P) \frac{\operatorname{sh}(\theta_1 - \theta_2)}{\operatorname{ch}\theta_1 \operatorname{ch}\theta_2} S_{ij}^{i'j'}(\theta_1 - \theta_2) \\
= (2\pi)^2 \delta^{(2)}(P' - P) \frac{4m^2 \operatorname{sh}(\theta_1 - \theta_2)}{4\varepsilon_1 \varepsilon_2} S_{ij}^{i'j'}(\theta_1 - \theta_2),
\end{aligned}
$$

where $P^\mu = p_1^\mu + p_2^\mu$, $P'^\mu = p_1'^\mu + p_2'^\mu$. Therefore, in the standard notation

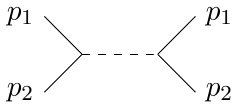
$$
S_{ij}^{i'j'}(\theta_1 - \theta_2) = \delta_i^{i'} \delta_j^{j'} + \frac{M_{ij}^{i'j'}(\theta_1 - \theta_2)}{4m^2 \operatorname{sh}(\theta_1 - \theta_2)},
$$

where the M_{ij} amplitude is calculated according the Feynman rules.

The compatibility condition with the $O(N)$-symmetry gives

$$
S_{ij}^{i'j'}(\theta) = \delta_{i'j'} \delta_{ij} S_1(\theta) + \delta_{i'i} \delta_{j'j} S_2(\theta) + \delta_{j'i} \delta_{i'j} S_3(\theta). \tag{16}
$$

In the order $1/N$, the matrix elements are given by the following diagrams:

$4m^2 \operatorname{sh}\theta \, S_1(\theta) =$ 

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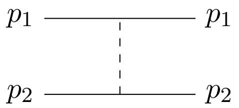
$4m^2 \operatorname{sh}\theta \, (S_2(\theta) - 1) =$ 

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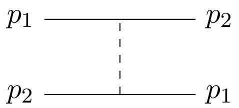
$4m^2 \operatorname{sh}\theta \, S_3(\theta) =$ 

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To calculate these diagrams, we need an explicit formula for $D(k)$. It has the form

$$
D^{-1}(k) = \frac{i}{2\pi k^2} \frac{1}{\sqrt{1 - \frac{4m^2}{k^2}}} \log \frac{\sqrt{1 - \frac{4m^2}{k^2}} + 1}{\sqrt{1 - \frac{4m^2}{k^2}} - 1}. \tag{17}
$$

This cumbersome formula becomes quite elementary in the parameterization

$$
k^2 = -4m^2 \operatorname{sh}^2 \frac{\theta}{2}. \tag{18}
$$

Note that the angle θ in this parameterization coincides with $\theta_1 - \theta_2$ in the case of the diagram for S_3. We have

$$
D(k) = 4\pi i m^2 \frac{\operatorname{sh}\theta}{\theta}. \tag{19}
$$

Substituting these expressions into the diagrams, we obtain

$$
\begin{aligned}
S_1(\theta) &= -\frac{2\pi i}{N(i\pi - \theta)}, \\
S_2(\theta) &= 1 - \frac{2\pi i}{N \operatorname{sh}\theta}, \\
S_3(\theta) &= -\frac{2\pi i}{N\theta}.
\end{aligned} \tag{20}
$$

Bibliography

[1] A. B. Zamolodchikov, Al. B. Zamolodchikov, *Annals Phys.* **120** (1979) 253.

[2] A. M. Polyakov, Gauge fields and strings, CRC Press, 1987.

[3] A. M. Tsvelik, Quantum field theory in condensed matter physics, Cambridge University Press, 2003.

Problems

1. Obtain formulas (17) and (19).

2. The Gross–Neveu model for the N-component Majorana (i.e. real in the representation with purely imaginary γ-matrices) Fermi-field is defined by the action

$$S[\psi] = \int d^2x \left(\frac{i}{2}\bar{\psi}_i \gamma^\mu \partial_\mu \psi_i + \frac{g}{8}(\bar{\psi}_i\psi_i)^2 \right)$$

(summation is assumed over repeated indices; in a representation with purely imaginary gamma-matrices we have $\bar{\psi} = \psi^T\gamma^0$).

Show that this model is equivalent to the model with an auxiliary boson field

$$S[\psi, \omega] = \int d^2x \left(\frac{1}{2}\bar{\psi}_i(i\gamma^\mu\partial_\mu - \omega(x))\psi_i - \frac{\omega^2(x)}{2g} \right).$$

Demonstrate that dynamic mass generation takes place in the model with

$$\omega_0 = m = \Lambda \exp\left(-\frac{2\pi}{Ng}\right).$$

3. Construct a diagrammatic technique for the $1/N$-decomposition in the Gross-Neveu model. Find the S-matrix in the tree approximation.

4. Consider the model with the weakened condition for $\boldsymbol{n}^2$:

$$S[\boldsymbol{n}] = \frac{1}{2g}\int d^2x \left((\partial_\mu \boldsymbol{n})^2 - \frac{2\mu^2}{g}(\boldsymbol{n}^2 - 1)^2 \right), \tag{21}$$

where μ is the constant with the dimension of mass. Find the mass of excitations in the model (21) in the leading order. Show that in the limit $\mu \to \infty$ the model tends to the sigma-model (1).

5*. Find the mass of excitation in the model from Problem 5 to Lecture 4 in the leading order in $1/N$.