# Lecture 11
# Kondo problem: derivation of the Bethe Ansatz

Beginning in the 1930s, a minimum was observed in the temperature dependence of the resistivity of some seemingly pure metals (gold, silver, copper) at low temperatures. Later it turned out that the anomaly is caused by the presence of a low concentration of impurity atoms of transition metals (Mn, Fe, Cr, Co, Ce, Y). Jun Kondo (1964) explained this phenomenon by electron scattering on impurities described by the interaction (*sd model*)

$$V = J \sum_i \boldsymbol{\sigma} \boldsymbol{S}_i \delta(\boldsymbol{r} - \boldsymbol{R}_i) \tag{1}$$

In the Born approximation, the scattering amplitude on a single impurity is proportional to

$$J(\boldsymbol{\sigma} \boldsymbol{S})_{\sigma'\sigma}.$$

Since $J$ is several times less than $\epsilon_F/n$ ($n$ is the electron density), this amplitude is small. However, the first correction to the Born amplitude contains logarithmic terms and the sum of the terms of the first and second orders in $J$ is[1]

$$J(\boldsymbol{\sigma} \boldsymbol{S})_{\sigma'\sigma} \left(1 + J\rho(\epsilon_F) \log \frac{\epsilon_F}{\max(|\epsilon_{\boldsymbol{p}} - \epsilon_F|, T)}\right), \tag{2}$$

where $\rho(\epsilon)$ is the density of states. This correction grows as $T \to 0$ and for electron energies close to $\epsilon_F$, and the perturbation theory eventually becomes unusable. The characteristic temperature, at which the perturbation theory is unusable, called the *Kondo temperature*, is equal to

$$T_K \sim \epsilon_F e^{-1/J\rho(\epsilon_F)}. \tag{3}$$

This temperature is the only characteristic scale in the Kondo effect.

Let us give a brief derivation of the expression (2). Double scattering can occur in two ways. First, an electron in the state $\boldsymbol{p}\sigma$ can pass first to the state $\boldsymbol{p}''\sigma''$, and then to the final state $\boldsymbol{p}'\sigma'$. The amplitude of this process is proportional to

$$J^2 \sum_{\sigma''} \int \frac{d^3 p''}{(2\pi)^3} \frac{(\boldsymbol{\sigma} \boldsymbol{S})_{\sigma'\sigma''}(\boldsymbol{\sigma} \boldsymbol{S})_{\sigma''\sigma}(1 - f(\boldsymbol{p}''))}{\epsilon_{\boldsymbol{p}} - \epsilon_{\boldsymbol{p}''}},$$

where $f(\boldsymbol{p})$ is the distribution function. Indices associated with the state of the impurity are omitted. Second, at first the electron from the filled state $\boldsymbol{p}''\sigma''$ passes into the state $\boldsymbol{p}'\sigma'$, and only then the electron from the state $\boldsymbol{p}\sigma$ passes into the state $\boldsymbol{p}''\sigma''$. The amplitude of this transition is proportional (with the same proportionality coefficient) to

$$-J^2 \sum_{\sigma''} \int \frac{d^3 p''}{(2\pi)^3} \frac{(\boldsymbol{\sigma} \boldsymbol{S})_{\sigma''\sigma}(\boldsymbol{\sigma} \boldsymbol{S})_{\sigma'\sigma''} f(\boldsymbol{p}'')}{\epsilon_{\boldsymbol{p}''} - \epsilon_{\boldsymbol{p}'}}.$$

The minus sign is related to the antisymmetry of the wave functions of electrons. If there were no spin factors, the contributions that contain $f(\boldsymbol{p}'')$ would be canceled, and there would be no logarithmically large factor. But for spin factors we have:

$$\sigma^i S^i \sigma^j S^j = S(S+1) - \boldsymbol{\sigma} \boldsymbol{S},$$
$$\sigma^i S^j \sigma^j S^i = S(S+1) + \boldsymbol{\sigma} \boldsymbol{S}.$$

We obtain the integral

$$J^2 \int \frac{d^3 p''}{(2\pi)^3} \left(\frac{S(S+1)\delta_{\sigma'\sigma}}{\epsilon_{\boldsymbol{p}} - \epsilon_{\boldsymbol{p}''}} + \frac{2f(\boldsymbol{p}'') - 1}{\epsilon_{\boldsymbol{p}} - \epsilon_{\boldsymbol{p}''}}(\boldsymbol{\sigma} \boldsymbol{S})_{\sigma'\sigma}\right).$$

The first term has no singularity at $\epsilon_{\boldsymbol{p}''} = \mu$ and produces a finite contribution near the Fermi surface. The second term produces a contribution that logarithmically diverges at zero temperature. It gives (2).

The resistivity in the first order is

$$\rho = \rho_v + \rho_J^{(0)} \left(1 + 2J\rho(\epsilon_F) \log \frac{\epsilon_F}{T}\right).$$

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In the case of ferromagnetic interaction $(J < 0)$, the contribution of magnetic impurities decreases with decreasing temperature, while in the case of antiferromagnetic interaction $(J > 0)$ it increases.

It turns out that one can sum up the logarithmic terms in all orders of the perturbation theory (Abrikosov, 1965; Suhl, 1965):

$$\rho = \rho_v + \frac{\rho_J^{(0)}}{\left(1 - J\rho(\epsilon_F)\log\frac{\epsilon_F}{T}\right)^2}.$$

However, for $T \sim T_K$ this expression has a singularity, so it is impossible to restrict oneself to the logarithmic corrections only.

Anomalies were also observed in the thermodynamic characteristics of metals with magnetic impurities, for example, in the heat capacity $C$ and the magnetic susceptibility $\chi$. At high temperatures, the impurity contribution to these quantities is described with good accuracy by the inverse powers of the logarithm:

$$\rho_{\text{imp}}(T) \simeq \frac{\text{const}}{\log^2\frac{T}{T_K}},$$

$$C_{\text{imp}}(T) \simeq \frac{\text{const}}{\log^2\frac{T}{T_K}},$$

$$\chi_{\text{imp}}(T) \simeq \frac{\text{const}}{T\log\frac{T}{T_K}}, \qquad T \gg T_K.$$

As $T \to 0$ a different behavior is observed:

$$\rho_{\text{imp}}(T) = \rho_{\text{imp}}(0)\left(1 - \kappa_R\left(\frac{T}{T_K}\right)^2 + \dots\right),$$

$$C_{\text{imp}}(T) = \gamma\frac{T}{T_K}\left(1 - \kappa_C\left(\frac{T}{T_K}\right)^2 + \dots\right),$$

$$\chi_{\text{imp}}(T) = \chi_0\left(1 - \kappa_\chi\left(\frac{T}{T_K}\right)^2 + \dots\right),$$

where $\kappa_R$, $\kappa_C$, $\kappa_\chi$ are quantities of order one.

The solution to this problem for $T \lesssim T_K$ seems rather hopeless. Nevertheless, to a certain approximation, this problem reduces to a problem that can be exactly solved by using the Bethe Ansatz (Wiegmann 1980; Andrei, 1980). There are good reviews[2, 3] on this subject.

Let us write down the Hamiltonian of the $sd$ model in the form

$$H = H_0 + J\boldsymbol{\sigma}(0)\boldsymbol{S}, \qquad H_0 = \sum_{\boldsymbol{p}\sigma}\epsilon_{\boldsymbol{p}}c^+_{\boldsymbol{p}\sigma}c_{\boldsymbol{p}\sigma}, \qquad \boldsymbol{\sigma}(0) = \sum_{\boldsymbol{p}'\sigma',\boldsymbol{p}\sigma}c^+_{\boldsymbol{p}'\sigma'}\boldsymbol{\sigma}_{\sigma'\sigma}c_{\boldsymbol{p}\sigma}. \tag{4}$$

It is assumed here that there is only one impurity in the system that interacts isotropically with free electrons. In addition, there is no potential impurity scattering. We will also assume that $\epsilon_{\boldsymbol{p}}$ does not depend on the direction of the momentum, and the spectrum near the Fermi-sphere has the form

$$\epsilon_{\boldsymbol{p}} = \epsilon_F + v_F(p - p_F). \tag{5}$$

Now we decompose the creation-annihilation operators into spherical functions:

$$c^+_{\boldsymbol{p}\sigma} = \sum_{lm}Y_{lm}(\boldsymbol{p}/p)c^+_{plm\sigma}. \tag{6}$$

Due to the orthogonality of spherical harmonics, the interaction Hamiltonian only contains the components with $l = m = 0$:

$$H = \sum_{plm\sigma}\epsilon_p c^+_{plm\sigma}c_{plm\sigma} + J\sum_{p'\sigma',p\sigma}c^+_{p'00\sigma'}c_{p00\sigma}\boldsymbol{\sigma}_{\sigma'\sigma}\boldsymbol{S}. \tag{7}$$

Electrons with $l > 0$ do not interact with electrons with $l = 0$ and with each other, and their contribution is trivial. The nontrivial part of the problem is to study the contribution of electrons with $l = 0$ to the

Hamiltonian. We will count the electron energy from the Fermi energy, and the quasimomentum from $p_F$. Besides, for simplicity we assume $v_F = 1$. We obtain

$$H = \sum_{p\sigma} p c_{p\sigma}^+ c_{p\sigma} + J \sum_{p'p\sigma'\sigma} c_{p'\sigma'}^+ c_{p\sigma} \boldsymbol{\sigma}_{\sigma'\sigma} \boldsymbol{S}, \tag{8}$$

where $c_{p\sigma} = c_{p_F+p,00\sigma}$, and the summation is taken over all real $p$. This Hamiltonian is one-dimensional. Moreover, in essence, this is the Hamiltonian defined on the entire material axis $x$, rather than on the semi-axis $x < 0$ only, since the reflected and incident waves do not interact with each other. In this case, however, waves in the system can only move in one (positive) direction. The cyclic boundary condition on the segment $[-R, R)$ will correspond to the condition for a wave that is reflected on a sphere of radius $R$ in the physical space. In the coordinate representation the Hamiltonian has the form

$$H = \int dx \left(-i c^+(x)\partial_x c(x) + J c^+(x)(\boldsymbol{\sigma}\boldsymbol{S})c(x)\delta(x)\right), \qquad c(x) = \begin{pmatrix} c_+(x) \\ c_-(x) \end{pmatrix} = \sum_p e^{ipx} \begin{pmatrix} c_{p+} \\ c_{p-} \end{pmatrix}. \tag{9}$$

The Hamiltonian $H$ preserves the number of particles in the system and the total spin. The state $|\Psi_N\rangle$ of the system with a fixed number of particles $N$ can be presented as

$$|\Psi_N\rangle = \int dx_1 \ldots dx_N \sum_{\sigma_1\ldots\sigma_N,s} \Psi^{\sigma_1\ldots\sigma_N,s}(x_1,\ldots,x_N)c_{\sigma_1}^+(x_1)\ldots c_{\sigma_N}^+(x_N)(S^-)^{S-s}|\Omega\rangle, \tag{10}$$

where $s = -S, -S+1\ldots, S$ is the value of the $z$-component of the spin $\boldsymbol{S}$, and $|\Omega\rangle$ is the pseudovacuum state satisfying the conditions

$$c_\sigma(x)|\Omega\rangle = S^+|\Omega\rangle = 0. \tag{11}$$

The Hamiltonian (9) acts on the state $|\Psi_N\rangle$ as follows:

$$\hat{H}\Psi^{\sigma_1\ldots\sigma_N,s} = -i\sum_{j=1}^N \partial_{x_j}\Psi^{\sigma_1\ldots\sigma_N,s} + J\sum_{j=1}^N \sum_{\sigma_j',s'} \delta(x_j)\boldsymbol{\sigma}_{\sigma_j\sigma_j'}\boldsymbol{S}_{ss'}\Psi^{\sigma_1\ldots\sigma_j'\ldots\sigma_N,s'} \tag{12}$$

We first consider a one-particle state. We will look for a solution in the form

$$\Psi_p^{\sigma,s}(x) = \begin{cases} A_p^{\sigma,s}e^{ipx}, & x < 0, \\ B_p^{\sigma,s}e^{ipx}, & x > 0. \end{cases} \tag{13}$$

Substituting (13) into (12), we obtain

$$A_p^{\sigma,s} = \sum_{\sigma',s'} R_{\sigma's'}^{\sigma s}B^{\sigma',s'}, \qquad R = e^{iJ\boldsymbol{\sigma}\boldsymbol{S}}. \tag{14}$$

Now, it would seem, it is possible to elementarily construct a wave function of an arbitrary state as the antisymmetrized product of wave functions of the form (13). This, however, is not the case. The fact is that the model has an additional dynamic variable $s$. Consider, for example, the region $x_1, x_2 < 0$. Suppose that in this region the wave function is continuous. We extend the wave function to the region $x_1 < 0 < x_2$. The 2 particle was scattered by impurities; therefore, the wave function in this region will differ in the scattering factor $R_{20}$. Continuing into the region $0 < x_1 < x_2$, we obtain the factor $R_{20}R_{10}$. Similarly, continuing first into the region $x_2 < 0 < x_1$, and then into the region $0 < x_2 < x_1$, we obtain the factor $R_{10}R_{20}$. But

$$R_{20}R_{10} \neq R_{10}R_{20}.$$

It follows that on the line $x_1 = x_2$ in the region $x_1, x_2 > 0$ the wave function has a discontinuity.

We will consider solutions with discontinuities at $x_1 = x_2$ both in the region $x_1, x_2 > 0$ and in the region $x_1, x_2 < 0$. Let us see which discontinuities are compatible with the Schrödinger equation. Let, for example, $x_1, x_2 < 0$. According to the Schrödinger equation, we have

$$(\partial_{x_1} + \partial_{x_2})\Psi^{\sigma_1\sigma_2,s}(x_1, x_2) = (\text{finite for } x_1 = x_2 \text{ expression}).$$

This condition admits arbitrary discontinuity of the wave function at $x_1 = x_2$. Indeed,

$$\partial_{x_1}\Psi(x_1, x_2) = \delta(x_1 - x_2)(\Psi(x_1 + 0, x_2) - \Psi(x_1 - 0, x_2)) + \ldots,$$
$$\partial_{x_2}\Psi(x_1, x_2) = \delta(x_1 - x_2)(\Psi(x_1, x_2 + 0) - \Psi(x_1, x_2 - 0)) + \ldots$$
$$= -\delta(x_1 - x_2)(\Psi(x_1 + 0, x_2) - \Psi(x_1 - 0, x_2)) + \ldots.$$

Thus you can choose any gap.

We will seek a solution to the Schrödinger equation in the form of Bethe Ansatz in the variables $(\sigma_1, x_1)$, ..., $(\sigma_N, x_N)$, $(s, 0)$, and the scattering matrix coefficients for the particle $j \neq 0$ with a particle $0$ (the impurity) will be given by the matrix $R$, and the scattering of two electrons $j, j' \neq 0$ will be given by some matrix $S$, which we will find from the associativity conditions (the Young–Baxter equations):

$$S_{12}R_{10}R_{20} = R_{20}R_{10}S_{12}. \tag{15}$$

An obvious solution to this equation is the permutation matrix

$$S_{12} = P_{12}, \qquad P_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'} = \delta_{\sigma_2}^{\sigma_1'}\delta_{\sigma_1}^{\sigma_2'}. \tag{16}$$

If the particles are denoted by solid lines, and the impurity by a dotted one, then graphically it looks like this:



In this notation, the proof of (15) for $S = P$ looks evident:



Now impose the cyclic boundary condition

$$\Psi(x_1, \ldots, x_j, \ldots, x_N) = \Psi(x_1, \ldots, x_j + L, \ldots, x_N). \tag{17}$$

Introduce the operator

$$T_j = P_{jj-1}\ldots P_{j1}R_{j0}P_{jN}\ldots P_{jj+1}. \tag{18}$$

From the cyclic boundary condition (17) it follows that

$$e^{ip_j L}\Psi = T_j\Psi. \tag{19}$$

This means that the solution to the problem is reduced to the simultaneous diagonalization of the operators $T_j$. It is easy to see that the operators $T_j$ coincide. Indeed, they can be written out as

$$T_j = T = \mathrm{tr}_{\tilde{1}}(P_{\tilde{1}N}\ldots P_{\tilde{1}1}R_{\tilde{1}0}), \tag{20}$$

where $\tilde{1}$ is the subscript for the auxiliary space. The additional operator $P_{\tilde{1}j}$ turns the product of operators into a trace.

How to diagonalize the transfer matrix $T$? To do this, you must reuse the Bethe Ansatz (the *secondary Bethe Ansatz*), and it is more convenient to use an algebraic Ansatz. But for this purpose it is necessary to build a family of commuting transfer matrices.

Let us construct one-parameter families of $R$-matrices $R(u)$, $S(u)$ that satisfy the following conditions.

1. The matrices $R(u)$ and $S(u)$ satisfy the Yang–Baxter equation:

$$S_{12}(u_1 - u_2)R_{10}(u_1 - u_0)R_{20}(u_2 - u_0) = R_{20}(u_2 - u_0)R_{10}(u_1 - u_0)S_{12}(u_1 - u_2), \tag{21a}$$

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$$S_{12}(u_1 - u_2)S_{13}(u_1 - u_3)S_{23}(u_2 - u_3) = S_{23}(u_2 - u_3)S_{13}(u_1 - u_3)S_{12}(u_1 - u_2). \tag{21b}$$

2. At special points, the matrices $S(u)$ and $R(u)$ coincide with $S$ and $R$:

$$S(0) = P, \qquad R(1) = R = e^{iJ\boldsymbol{\sigma}\boldsymbol{S}}. \tag{22}$$

3. $R$-matrices satisfy the unitarity condition:

$$S_{12}(u)S_{21}(-u) = 1, \qquad R_{10}(u)R_{10}(-u) = 1. \tag{23}$$

The solution can be represented as

$$\begin{aligned} S_{12}(u) &= w_0(u) + w(u)\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2, \\ R_{10} &= w_0'(u) + 2w'(u)\boldsymbol{\sigma}_1\boldsymbol{S}_0. \end{aligned} \tag{24}$$

It is convenient to introduce the notation

$$\begin{aligned} a &= w_0 + w, & b &= w_0 - w, & c &= 2w, \\ a' &= w_0' + w', & b' &= w_0' - w', & c' &= 2w'. \end{aligned} \tag{25}$$

In this case, the matrix $S(u)$ has the same form as the $R$-matrix of the XXZ model:

$$S(u) = \begin{pmatrix} a(u) & & & \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ & & & a(u) \end{pmatrix}.$$

By solving the Young–Baxter equation, we find

$$\begin{aligned} \frac{b(u)}{a(u)} &= \frac{b'(u)}{a'(u)} = \frac{u}{u + ig}, \\ \frac{c(u)}{a(u)} &= \frac{c'(u)}{a'(u)} = \frac{ig}{u + ig}, \end{aligned} \tag{26}$$

i.e. $S(u)$ coincides (up to a change in the scale of the spectral parameter and some simple matrix transformation that changes signs) with the $R$-matrix of the XXX model.

The unitarity condition requires that

$$a(u)a(-u) = 1, \qquad a'(u)a'(-u) = \frac{g^2 + u^2}{g^2(S + 1/2)^2 + u^2}. \tag{27}$$

Finally, the condition (22) gives

$$a(0) = 1, \qquad a'(1) = \frac{1 + ig}{2}(e^{iJS} + e^{-iJ(S+1)}) \tag{28}$$

and

$$g = \frac{1}{S + 1/2}\,\mathrm{tg}\,J(S + 1/2). \tag{29}$$

Otherwise, $a(u)$, $a'(u)$ are arbitrary functions.

Further, the one-parameter family of transfer matrices has the form

$$T(u) = \mathrm{tr}_{\tilde{1}}\, L_{\tilde{1}}(u), \qquad L_{\tilde{1}}(u) = S_{\tilde{1}N}(u)\dots S_{\tilde{1}1}(u)R_{\tilde{1}0}(u + 1), \tag{30}$$

and

$$T(0) = T, \qquad [T(u), T(v)] = 0. \tag{31}$$

$L$-operators satisfy the relations

$$S_{\tilde{1}\tilde{2}}(u_1 - u_2)L_{\tilde{1}}(u_1)L_{\tilde{2}}(u_2) = L_{\tilde{2}}(u_2)L_{\tilde{1}}(u_1)S_{\tilde{1}\tilde{2}}(u_1 - u_2). \tag{32}$$

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As in the case of the XXZ model, the $L$-operator can be presented as

$$L(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \tag{33}$$

while $A(u), \ldots, D(u)$ satisfy the relations of the same form as in the case of the XXZ model, but with different $a(u)$, $b(u)$, $c(u)$. It follows that the solution will be substantially the same.

Namely, the pseudo-vacuum $|\Omega_N\rangle$, corresponding to all electron spins looking up ($\sigma_j = +$), and $s = +S$, is defined by the condition

$$C(u)|\Omega_N\rangle = 0. \tag{34}$$

The Bethe Ansatz has the form

$$|u_1, \ldots, u_n\rangle = B(u_1) \ldots B(u_n)|\Omega_N\rangle, \qquad S^z = N/2 + S - n, \tag{35}$$

with

$$\begin{aligned}
A(u)|\Omega_N\rangle &= \Lambda_A(u)|\Omega_N\rangle, & \Lambda_A(u) &= ((S+1/2)a'(u+1) - (S-1/2)b'(u+1))a^N(u), \\
D(u)|\Omega_N\rangle &= \Lambda_D(u)|\Omega_N\rangle, & \Lambda_D(u) &= ((S+1/2)b'(u+1) - (S-1/2)a'(u+1))b^N(u).
\end{aligned} \tag{36}$$

The Bethe equations are written in the standard form

$$\frac{\Lambda_D(u_i)}{\Lambda_A(u_i)} = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{a(u_j - u_i)b(u_i - u_j)}{b(u_j - u_i)a(u_i - u_j)}, \tag{37}$$

while the eigenvalues are expressed in terms of the roots of the Bethe equations as

$$\Lambda(u; u_1, \ldots, u_N) = \Lambda_A(u) \prod_{i=1}^{n} \frac{a(u_i - u)}{b(u_i - u)} + \Lambda_D(u) \prod_{i=1}^{n} \frac{a(u - u_i)}{b(u - u_i)}. \tag{38}$$

Equation (19) takes the form

$$e^{ip_j L} = \Lambda(0; u_1, \ldots, u_N). \tag{39}$$

By substituting explicit formulas for $a(u)$ and $b(u)$ into (37) and (38), (39), and replacing

$$u_j = g(v_j - i/2),$$

we obtain the Bethe equations in the form

$$\left(\frac{v_i + i/2}{v_i - i/2}\right)^N \frac{v_i + iS + g^{-1}}{v_i - iS + g^{-1}} = -\prod_{j=1}^{n} \frac{v_i - v_j + i}{v_i - v_j - i}, \tag{40}$$

and

$$e^{ip_j L} = e^{iJS} \prod_{i=1}^{n} \frac{v_i + i/2}{v_i - i/2}. \tag{41}$$

This reduces the solution of the Kondo problem to the joint solution of the equations (40) and (41).

## Bibliography

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## Problems

**1.** Derive (14).

**2.** Show that the transfer matrices $T(u)$ defined by (30) indeed form a family of commuting transfer matrices.

**3.** Show that the matrices $S(u)$, $R(u)$ defined in (24)—(26) satisfy the Yang–Baxter equations (21).

**4.** Derive the relations (27)—(29).

**5**$^*$**.** Construct solutions to the equations (21) with a trigonometric dependence on the spectral parameter $u$. Show that such solutions correspond to the Kondo problem with anisotropic interaction between the electrons and the impurity.