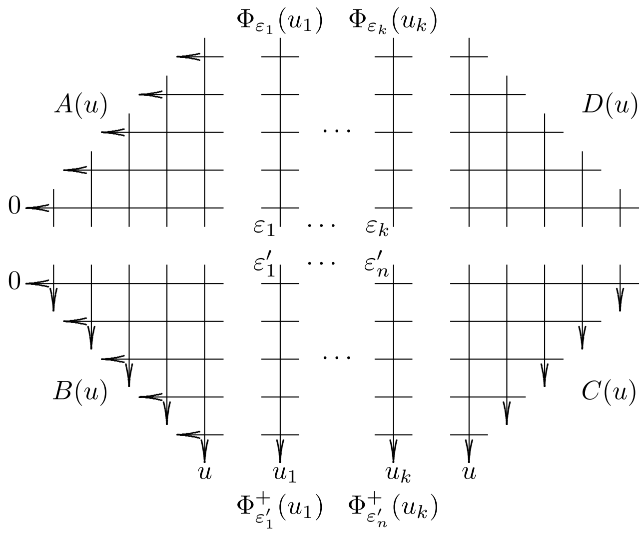
Lecture 3
Corner transfer matrices and vertex operators

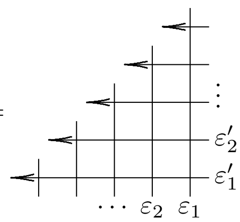
Consider the eight-vertex model on a large but finite lattice with fixed boundary condition. Set the spectral parameter on the horizontal lines to be equal to 0, while on the vertical lines to be equal to u always except k neighboring lines, where it will be equal to $u_1, \ldots, u_k$. Let us partition the lattice into several pieces as follows:



As it is shown in the picture at each of the k exceptional lines we indeed cut the bond in the very middle and fix the variables ε_i, ε'_i at the ends.

The pieces $A(u)$, $B(u)$, $C(u)$, $D(u)$ can be considered as matrices acting clockwise, e.g.

$A(u)^{\varepsilon'_1\varepsilon'_2\ldots}_{\varepsilon_1\varepsilon_2\ldots} =$



The matrices $A(u)$, $B(u)$, $C(u)$, $D(u)$ are called *corner transfer matrices.*

The pieces $\Phi_{\varepsilon_i}(u_i)$ are $\Phi^+_{\varepsilon_i}(u_i)$ act as matrices in the left-to-right and right-to-left directions respectively. They are called *half transfer matrices* and (because of some properties in the infinite volume limit) *vertex operators.*

We have to specify the boundary conditions at the outer boundary. We shall fix the spins at the boundary so as if they belong to one of the ground states described in the first lecture. We shall denote the boundary condition by the superscript (i) $(i \in \mathbb{Z}_2)$, if $\varepsilon_1 = (-1)^i$ in the corresponding ground state. To avoid multiple usage of this superscript at any corner transfer matrix and vertex operator like $A^{(i)}$, $B^{(i)}$, $C^{(i+n)}$, $D^{(i+n)}$, $\Phi^{(i+j,i+j-1)}_{\varepsilon_j}(u_j)$, we shall put it at the trace signs below.

Let $Z^{(i)}{}^{\varepsilon_1\ldots\varepsilon_k}_{\varepsilon'_1\ldots\varepsilon'_k}$ be the partition function of the lattice with the given boundary conditions and fixed variables at the upper and lower banks of the cut. Let $Z^{(i)} = \sum_{\varepsilon_1\ldots\varepsilon_k} Z^{(i)}{}^{\varepsilon_1\ldots\varepsilon_k}_{\varepsilon_1\ldots\varepsilon_k}$ be the partition functions of the lattice without the cut. Now consider the ratio $P^{(i)}{}^{\varepsilon_1\ldots\varepsilon_k}_{\varepsilon'_1\ldots\varepsilon'_k} = Z^{(i)}{}^{\varepsilon_1\ldots\varepsilon_k}_{\varepsilon'_1\ldots\varepsilon'_n}/Z^{(i)}$. In particular, $P^{(i)}{}^{\varepsilon_1\ldots\varepsilon_k}_{\varepsilon_1\ldots\varepsilon_k}$ is the probability that the configuration of spins on the bonds in the middle of the exceptional lines is $\varepsilon_1, \ldots, \varepsilon_k$.

These quantities are basic for calculation of local correlation functions. For example, let us calculate the average $\langle\varepsilon_1\varepsilon_2\rangle \equiv \langle\sigma^z_1\sigma^z_2\rangle$ of the product of two neighboring spins on the lattice without any cut. It is given by

$$\langle\sigma^z_1\sigma^z_2\rangle^{(i)} = P^{(i)}{}^{++}_{++} + P^{(i)}{}^{--}_{--} - P^{(i)}{}^{+-}_{+-} - P^{(i)}{}^{-+}_{-+}.$$

Other local correlation functions are expressed similarly.

From the partition of the lattice described above it is easy to obtain

$$P^{(i)\varepsilon_1\ldots\varepsilon_k}_{\varepsilon'_1\ldots\varepsilon'_n} = \frac{1}{Z^{(i)}}\operatorname{Tr}^{(i)}(\Phi^+_{\varepsilon'_1}(u_1)\ldots\Phi^+_{\varepsilon'_n}(u_k)C(u)D(u)\Phi_{\varepsilon_k}(u_k)\ldots\Phi_{\varepsilon_1}(u_1)A(u)B(u)). \tag{1}$$

Surely, we have not yet approached the exact solution to the problem. Nevertheless, in the large volume limit the objects defined above get remarkable properties.

First of all, not all of these objects are independent. From the crossing symmetry it is easy to find that

$$C(u) = QA(u)Q, \qquad B(u) = QD(u)Q = A(1-u)Q, \qquad \Phi^+_\varepsilon(u) = Q\Phi_\varepsilon(u)Q, \tag{2}$$

where $Q = \sigma^x \otimes \sigma^x \otimes \ldots$ is the operator that flips all spins.

Baxter observed [1] that in the large volume limit

$$A(u) = F(u)e^{-\epsilon uH} \tag{3}$$

with some scalar (not operator) function $F(u)$ and some constant operator H with the discrete spectrum $\{0, 1, 2, \ldots\}$.

Let us sketch Baxter's argumentation. Consider the product $A(u)B(u-v)$. On the infinite lattice this product, considered as a vector, must be an eigenvector corresponding to the largest eigenvalue of the transfer matrix of an inhomogeneous model. Hence, the product $A(u)A(1+v-u)$ as a function of u is a constant operator times a scalar function. As $A(0) = 1$, we have the equation

$$A(u)A(u') = g(u, u')A(u+u'). \tag{4}$$

Therefore

$$A(u)A'(0) = g_{u'}(u, 0)A(u) + g(u, 0)A'(u)$$

or

$$\frac{A'(u)}{A(u)} = f_1(u)A'(0) + f_2(u).$$

Solving this differential equation we obtain

$$A(u) = F_1(u)e^{F_2(u)A'(0)},$$

with some functions $F_1(u)$, $F_2(u)$, such that

$$F_1(0) = 1, \qquad F_2(0) = 0, \qquad F'_2(0) = 1.$$

Substituting this solution back into the difference equation (4), we obtain that $F_2(u) = u$. We obtain

$$A(u) = F_1(u)e^{uA'(0)}. \tag{5}$$

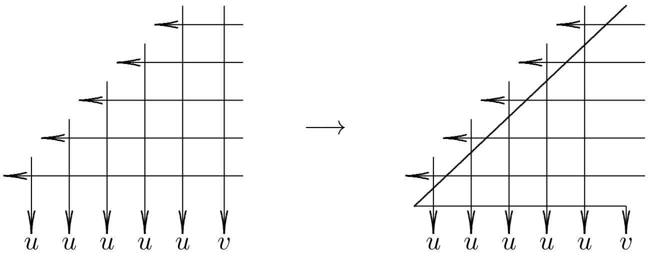
From the definition of the model we know that the Boltzmann weight without the function $\rho(u)$ are doubly periodic. They are unchanged after the substitutions $u \to u+2r$ and $u \to u+2i\pi/\epsilon$. The function $\rho(u)$ written out in the first lecture do not respect the first periodicity and respects the second. This is very important. The function $\rho(u)$ can be obtained directly from the Bethe ansatz, i.e. from first principles. Moreover, it can be concluded from the Bethe ansatz solution that all physical quantities respect this second periodicity. Therefore, impose this periodicity on the solution (5). We immediately obtain that the spectrum of $A'(u)$ is equidistant with the separation ϵ. This 'proves' (on the physical level of rigorousness) the equation (3). From now on we shall omit the factor $F(u)$ and write

$$A(u) = e^{-\epsilon uH} = z^{H/2}. \tag{6}$$

Now, on the physical level of rigorousness, it is easy to obtain the following two commutation relations [2]:

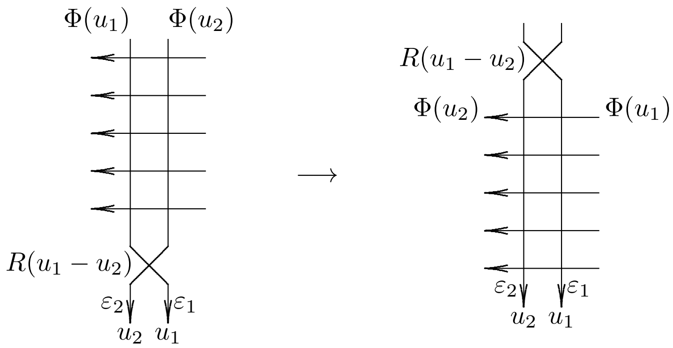
$$\begin{aligned}
\Phi_\varepsilon(u)Q &= Q\Phi_{-\varepsilon}(u),\\
\Phi_\varepsilon(v)A(u) &= A(u)\Phi_\varepsilon(v-u),\\
\Phi_{\varepsilon_1}(u_1)\Phi_{\varepsilon_2}(u_2) &= \sum_{\varepsilon'_1\varepsilon'_2} R(u_1-u_2)_{\varepsilon_1\varepsilon_2}^{\varepsilon'_1\varepsilon'_2}\Phi_{\varepsilon'_2}(u_2)\Phi_{\varepsilon'_1}(u_1).
\end{aligned}\tag{7}$$

The first equation is trivial. The second equation can be proven as follows. Take the product $\Phi_\varepsilon(v)A(u)$ and, using the Yang–Baxter equation, push the line corresponding to Φ_ε to the left:



Any physicist knows that the boundary does not affect essentially the bulk. So let us forget completely about the skew line on the right picture. Its only effect is the boundary condition. The lower horizontal line is just the operator $\Phi_\varepsilon(u-v)$.

The derivation of the last line is similar. Consider the product in the r.h.s. and push the R matrix upside:



Now the R matrix 'at the infinity' can be erased and we obtain the third equation.

From these equations we derive that

$$\Phi^+_\varepsilon(u_j)C(u)D(u) = Q\Phi_{-\varepsilon}(u_j)e^{-2\epsilon H} = Qe^{-2\epsilon H}\Phi_{-\varepsilon}(u_j-1) = C(u)D(u)\Phi_{-\varepsilon}(u_j-1).$$

Let us introduce the notation

$$\Phi^*_\varepsilon(u) = \Phi_{-\varepsilon}(u-1).\tag{8}$$

Then

$$\Phi^+_\varepsilon(u_j)C(u)D(u) = C(u)D(u)\Phi^*_\varepsilon(u_j).\tag{9}$$

It means that we can move the product $C(u)D(u)$ to the left simultaneously replacing $\Phi^+_\varepsilon(u_j)$ by $\Phi^*_\varepsilon(u_j)$. From the fact that $\sum_\varepsilon \Phi_\varepsilon(u)\otimes(\Phi^+_\varepsilon(u))^t$ is just the transfer matrix on the infinite lattice with the largest eigenvalue 1 we conclude that

$$\sum_\varepsilon \Phi^*_\varepsilon(u)\Phi_\varepsilon(u) = 1, \qquad \Phi_\varepsilon(u)\Phi^*_{\varepsilon'}(u) = \delta_{\varepsilon\varepsilon'}.\tag{10}$$

Now consider the product of the product of the corner transfer matrices. It is easy to find from (6) that

$$A(u)B(u)C(u)D(u) = e^{-4\epsilon H} = x^{2H}.\tag{11}$$

We can also specify what we mean under $\mathrm{Tr}^{(i)}$ in the infinite volume limit. Consider the consequences of spin variables $\varepsilon(1), \varepsilon(2), \ldots$ that stabilize to $\varepsilon(n) = (-)^{i+n}$. The space of such *paths* will be denoted by $\mathcal{H}^{(i)}$. Then $\mathrm{Tr}^{(i)} = \mathrm{Tr}_{\mathcal{H}^{(i)}}$.

Substituting (9) and (11) to (1) we obtain

$$P^{(i)}{}^{\varepsilon_1 \ldots \varepsilon_k}_{\varepsilon'_1 \ldots \varepsilon'_n} = \frac{1}{\chi^{(i)}} \mathrm{Tr}_{\mathcal{H}^{(i)}} (\Phi^*_{\varepsilon'_1}(u_1) \ldots \Phi^*_{\varepsilon'_n}(u_k) \Phi_{\varepsilon_k}(u_k) \ldots \Phi_{\varepsilon_1}(u_1) x^{2H}). \tag{12}$$

with

$$\chi^{(i)} = \mathrm{Tr}_{\mathcal{H}^{(i)}} \, x^{2H}. \tag{13}$$

It was shown [3], that

$$\mathrm{Tr}_{\mathcal{H}^{(i)}} \, q^H = \frac{1}{(q;q^2)_\infty}.$$

This result was proven in the limit $x \to 0$, $r \to \infty$, but, since degeneracy cannot change continuously, it holds in the whole AF_1 phase.

Generally, we can consider the *trace functions*

$$F^{(i)}_{\varepsilon_1 \ldots \varepsilon_k}(u_1, \ldots, u_k) = \frac{1}{\chi^{(i)}} \mathrm{Tr}_{\mathcal{H}^{(i)}} (\Phi_{\varepsilon_k}(u_k) \ldots \Phi_{\varepsilon_1}(u_1) x^{2H}). \tag{14}$$

Evidently,

$$P^{(i)}{}^{\varepsilon_1 \ldots \varepsilon_k}_{\varepsilon'_1 \ldots \varepsilon'_n} = F^{(i)}_{\varepsilon_1, \ldots, \varepsilon_k, -\varepsilon'_n, \ldots, -\varepsilon'_1}(u_1, \ldots, u_k, u_k - 1, \ldots, u_1 - 1).$$

The functions $F^{(i)}(u_1, \ldots, u_k)$ satisfy a number of difference equations, that follow from the properties of the corner transfer matrices and vertex operators:

$$F^{(i)}_{\varepsilon_1 \ldots \varepsilon_k}(u_1 + v, \ldots, u_k + v) = F^{(i)}_{\varepsilon_1 \ldots \varepsilon_k}(u_1, \ldots, u_k), \tag{15}$$

$$F^{(i)}_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_k}(u_1 + 2\mathrm{i}\pi/\epsilon, u_2, \ldots, u_k) = F^{(i)}_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_k}(u_1, u_2, \ldots, u_k), \tag{16}$$

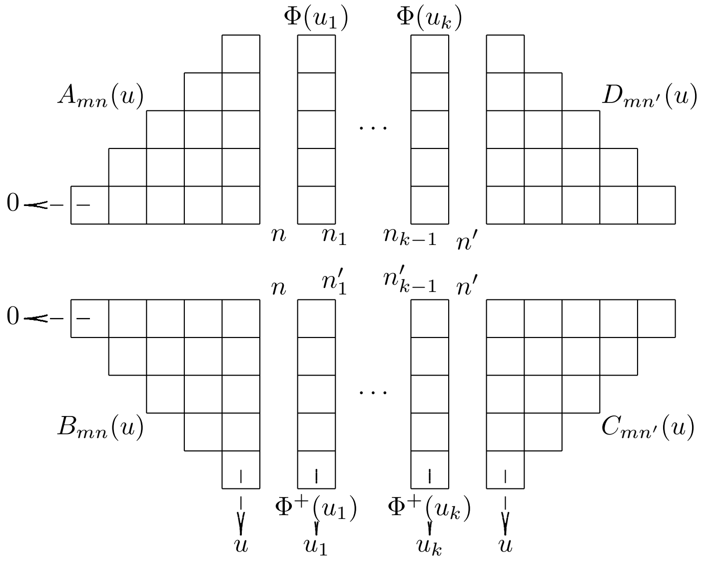
$$F^{(i)}_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_k}(u_1, u_2, \ldots, u_k) = F^{(i+1)}_{\varepsilon_2 \ldots \varepsilon_k \varepsilon_1}(u_2, \ldots, u_k, u_1 - 2), \tag{17}$$

$$\sum_{\varepsilon} F^{(i)}_{\varepsilon_1, \ldots, \varepsilon_k, \varepsilon, -\varepsilon}(u_1, \ldots, u_k, u, u-1) = F^{(i)}_{\varepsilon_1 \ldots \varepsilon_k}(u_1, \ldots, u_k), \tag{18}$$

$$F^{(i)}_{\ldots \varepsilon_j \varepsilon_{j+1} \ldots}(\ldots, u_j, u_{j+1}, \ldots) = \sum_{\varepsilon'_j \varepsilon'_{j+1}} R(u_{j+1} - u_j)^{\varepsilon'_j \varepsilon'_{j+1}}_{\varepsilon_j \varepsilon_{j+1}} F^{(i)}_{\ldots \varepsilon'_{j+1} \varepsilon'_j \ldots}(\ldots, u_{j+1}, u_j, \ldots). \tag{19}$$

In principle, it is possible to find the functions F and probabilities P by solving these equations under some analyticity conditions. In practice, the case $n = 2$ in F only admits a direct solution. In more general case we need some additional ideas to solve the equations.

Consider now the SOS model. The necessary partition of the lattice looks like:



Here we introduce the corner transfer matrices $A_{mn}(u), \ldots, D_{mn}(u)$, which depend on the 'central site' variable n and the boundary condition $(m, m+1)$ or $(m+1, m)$ depending on the parity of $n-m$, and the vertex operators $\Phi(u_j)^{n_j}_{n_{j-1}}$ and $\Phi^+(u_j)^{n'_{j-1}}_{n'_j}$, which depend on the variables at their feet.

In the same way as for the eight-vertex model one can write the quantities

$$P_m\left(n^{n_1\ldots n_{k-1}}_{n'_1\ldots n'_{k-1}}n'\right) = \frac{1}{Z_m}\,\mathrm{Tr}(\Phi^+(u_1)^{n_1}_{n'_1}\ldots\Phi^+(u_k)^{n'_{k-1}}_{n'}C_{mn}(u)D_{mn}(u)$$
$$\times\,\Phi(u_k)^{n'}_{n_{k-1}}\ldots\Phi(u_1)^{n_1}_{n}A_{mn}(u)B_{mn}(u)). \quad (20)$$

In particular, for $n'_i = n_i$ $(i = 1, \ldots, k-1)$ they are the multipoint *local height probabilities*, which describe the probabilities of configurations along a finite line on the lattice.

In the infinite volume limit we have up to a scalar factor

$$A_{mn}(u) = e^{-2\epsilon u H_{mn}}. \quad (21)$$

The product of the corner transfer matrices is given by

$$A_{mn}(u)B_{mn}(u)C_{mn}(u)D_{mn}(u) = [n]x^{4H_{mn}}. \quad (22)$$

The additional factor 2 before the corner Hamiltonian in comparison with the eight-vertex model is related to the (quasi)periodicity of all quantities with the period $\mathrm{i}\pi/\epsilon$ instead of $2\pi\mathrm{i}/\epsilon$, while the factor $[n]$ in the product is related to the similar factors in the crossing property.

Another important property is

$$\Phi^+(u)^{n'}_{n}C_{mn}(v)D_{mn}(v) = C_{mn'}(v)D_{mn'}(v)\Phi^*(u)^{n'}_{n}, \quad (23)$$

where

$$\Phi^*(u)^{n'}_{n} = [n]\Phi(u-1)^{n'}_{n}, \qquad \sum_{n'}\Phi^*(u)^{n}_{n'}\Phi(u)^{n'}_{n} = 1, \qquad \Phi(u)^{n'}_{n''}\Phi^*(u)^{n''}_{n} = \delta^{n'}_{n}. \quad (24)$$

The basic commutation relations look like

$$\Phi(u)^{n'}_{n}x^{2vH_{mn}} = x^{2vH_{mn'}}\Phi(u-v)^{n'}_{n}, \quad (25)$$

$$\Phi(u_1)_s^{n'}\Phi(u_2)_n^s = \sum_{s'} W\begin{bmatrix} n & s' \\ s & n' \end{bmatrix} u_1 - u_2 \Bigg] \Phi(u_2)_{s'}^{n'}\Phi(u_1)_n^{s'}. \tag{26}$$

In the infinite volume limit define the trace functions

$$F^m_{nn_1\ldots n_k}(u_1,\ldots,u_k) = \frac{[n]}{\chi_m}\operatorname{Tr}_{\mathcal{H}_{mn}}(\Phi(u_k)^n_{n_k}\ldots\Phi(u_2)^{n_2}_{n_1}\Phi(u_1)^{n_1}_n x^{4H_{mn}}) \tag{27}$$

with

$$\chi_m = \sum_n [n]\chi_{mn}, \qquad \chi_{mn} = \operatorname{Tr}_{\mathcal{H}_{mn}} x^{4H_{mn}}. \tag{28}$$

Here $\mathcal{H}_{mn}$ is the space of paths $n(0)=n, n(1), n(2), \ldots$ that stabilize to the sequence $\ldots, m, m+1, m, m+1, \ldots$. There is a remarkable result by Andrews, Baxter and Forrester [4] based on the $x\to 0$ limit (the *low temperature* limit) that

$$\operatorname{Tr}_{\mathcal{H}_{mn}} q^{H_{mn}} = \frac{q^{(rm-(r-1)n)^2/4r(r-1)}}{(q;q)_\infty}.$$

Notice an important property of χ_{mn}:

$$\sum_{n\in 2\mathbb{Z}+m+i}[n]\chi_{mn} = [m]'\chi^{(i)}, \qquad [u]' = [u]|_{r\to r-1}. \tag{29}$$

From the properties (22) and (23) we obtain

$$P_m\begin{pmatrix} n\, {n_1\ldots n_{k-1} \atop n'_1\ldots n'_{k-1}}\, n' \end{pmatrix} = F^m_{nn_1\ldots n_{k-1}n'n'_{k-1}\ldots n'_1}(u_1,\ldots,u_k,u_k-1,\ldots,u_1-1)[n']\prod_{j=1}^{k-1}[n'_k].$$

Using the commutation relations (25), (26) we obtain

$$F^m_{nn_1\ldots n_{k-1}}(u_1+v,\ldots,u_k+v) = F^m_{nn_1\ldots n_{k-1}}(u_1,\ldots,u_k), \tag{30}$$

$$F^m_{nn_1n_2\ldots n_{k-1}}(u_1+\mathrm{i}\pi/\epsilon,u_2,\ldots,u_k) = -e^{\mathrm{i}\pi(n_1^2-n^2)/2r}F^m_{nn_1n_2\ldots n_k}(u_1,u_2,\ldots,u_k), \tag{31}$$

$$F^m_{nn_1n_2\ldots n_k}(u_1,u_2,\ldots,u_k) = \frac{[n]}{[n_1]}F^m_{n_1n_2\ldots n_kn}(u_2,\ldots,u_k,u_1), \tag{32}$$

$$\sum_{n'}[n']F^m_{nn_1\ldots n_{k-1}nn'}(u_1,\ldots,u_k,u,u-1) = F^m_{nn_1\ldots n_{k_1}}(u_1,\ldots,u_k), \tag{33}$$

$$F^m_{\ldots n_{k-1}n_kn_{k+1}\ldots}(\ldots,u_k,u_{k+1},\ldots) = \sum_{n'_k} W\begin{bmatrix} n_{k-1} & n'_k \\ n_k & n_{k+1} \end{bmatrix} u_{k+1}-u_k\Bigg] F^m_{\ldots n_{k-1}n'_kn_{k+1}\ldots}(\ldots,u_{k+1},u_k,\ldots). \tag{34}$$

The problem of solving these equations in the SOS model will be discussed in the next lecture. The respective problem for the eight-vertex model is more difficult and will be the topic of the last lecture.

References

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[2] M. Jimbo, T. Miwa, *Algebraic analysis of solvable lattice models*, CBMS Regional Conference Series in Mathematics **85** AMS (1994)

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