Lecture 5
Conformal field theory: Virasoro algebra and its representations

Let us try to understand better the Virasoro algebra:

$$[L_k, L_l] = (k-l)L_{k+l} + \frac{c}{12}k(k^2-1)\delta_{k+l,0}. \tag{1}$$

First of all, let $c = 0$. In this case the algebra can be realized in terms of differential operators

$$L_k = -z^{k+1}\partial.$$

An infinitesimal action of a function $F(z)$ looks like

$$-\sum \epsilon_k L_k F(z) = F\left(z + \sum \epsilon_k z^{k+1}\right) = F(z+\epsilon(z)). \tag{2}$$

The function $z + \epsilon(z)$ is a small conformal transformation. Hence, we may conclude that the Virasoro algebra generates conformal transformations. But not every conformal transformation is a transformation of the whole plane. Most of them have singularities, and even not single-valued function. Let us restrict ourselves to transformation that are single-value and have not more than one single pole, including pole at infinity. More precisely, we will consider single-valued conformal transformation of the sphere $\mathbb{C}\cup\{\infty\}$. These are Möbius transformations:

$$f(z) = \frac{az+b}{cz+d}. \tag{3}$$

Any Möbius transformation is a composition of three basic transformations:

1. Translation: $f(z) = z + b$;
2. Dilation: $f(z) = az$;
3. Inversion: $f(z) = z^{-1}$.

The translation and dilation transformation are evidently generated by L_{-1} and L_0 correspondingly. But the inversion cannot be infinitesimal. The corresponding infinitesimal transformation is the composition of two inversions with a translation in between:

$$f(z) = (z^{-1}-\epsilon)^{-1} = \frac{z}{1-\epsilon z} = z + \epsilon z^2 + O(c^2).$$

We see that this transformation is generated by the operator $-cL_1$. Hence the triple of operators L_{-1}, L_0, L_1 generate the Möbius transformations. Independently of the central charge this algebra forms an $sl(2)$ subalgebra

$$[L_1, L_{-1}] = 2L_0, \qquad [L_{\pm1}, L_0] = \pm L_{\pm1}. \tag{4}$$

All other operators L_k correspond to transformations that are non-single-valued on the sphere and acquire an anomaly, which expresses itself as the central charge.

Consider the action of the Virasoro algebra (4) on an operator. Let

$$\delta_\epsilon \Phi(z,\bar z) = \left[\sum_k \epsilon_k L_k, \Phi(z,\bar z)\right]. \tag{5}$$

For a primary operator Φ with conformal dimensions $(\Delta, \bar\Delta)$ we have

$$\begin{aligned}\delta_\epsilon \Phi(z,\bar z) &= \sum_k (k+1)\epsilon_k z^k \Delta\Phi(z,\bar z) + \sum_k \epsilon_k z^k \partial\Phi(z,\bar z)\\ &= \Delta\epsilon'(z)\Phi(z,\bar z) + \Phi(z+\epsilon(z)) - \Phi(z,\bar z) = ((z+\epsilon(z))')^{\Delta}\Phi(z+\epsilon(z),\bar z) - \Phi(z,\bar z).\end{aligned}$$

Taking into account the second chiral part and integrating the transformation we obtain

$$\Phi(z,\bar z) \to (f'(z))^{\Delta}(\bar f'(\bar z))^{\bar\Delta}\Phi(f(z),\bar f(\bar z)). \tag{6}$$

If $f(z)$ and $\bar f(\bar z)$ are both Möbius transformations, the correlation function must be invariant under this transformation. For general functions $f(z), \bar f(\bar z)$ this transforms different surfaces. In particular,

$$f(z) = \mathrm{i}\frac{L}{2\pi}\log z, \qquad \bar f(\bar z) = -\mathrm{i}\frac{L}{2\pi}\log z \tag{7}$$

maps the Euclidean plane onto the Euclidean cylinder, as we have discussed.

It is important that the operator $T(z)$, being a dimensions $(2,0)$ operator, is not primary. Indeed, its operator product expansion

$$T(z')T(z) = \frac{c/2}{(z'-z)^4} + \frac{2T(z)}{(z'-z)^2} + \frac{\partial T(z)}{z'-z} + O(1) \tag{8}$$

contains the central charge term, which is not allowed for a primary operator. We may rewrite it as

$$[L_k, T(z)] = \frac{c}{12}k(k^2-1)z^{k-2} + 2(k+1)z^k T(z) + z^{k+1}\partial T(z). \tag{9}$$

We easily obtain

$$\delta_\epsilon T(z) = \frac{c}{12}\epsilon'''(z) + 2\epsilon'(z)T(z) + \epsilon(z)\partial T(z). \tag{10}$$

Integration of this transformation is not so easy, but it results:

$$T(z) \to (f'(z))^2 T(f(z)) + \frac{c}{12}\{f(z), z\}, \quad \{f(z), z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2. \tag{11}$$

The last bracket is called Schwartz derivative. For the transformation (7) we have $\{f(z), z\} = 1/2z^2$, which gives

$$T(\zeta) = -\left(\frac{2\pi}{L}\right)^2\left(z^2 T(z) - \frac{c}{24}\right). \tag{12}$$

Hence,

$$H_{\mathrm{cyl}} = \frac{2\pi}{L}\left(L_0 + \bar L_0 - \frac{c}{12}\right). \tag{13}$$

This gives the vacuum energy in the cylinder geometry:

$$E_{\mathrm{vac}} = -\frac{\pi c}{6L}, \tag{14}$$

which is consistent with the free boson and free fermion cases. The advance of this derivation is that it involves no regularization nor renormalization.

Now let us study highest weight representation. Let $|\Delta\rangle$ be the highest weight vector of the Virasoro algebra:

$$L_k|\Delta\rangle = 0 \quad (k>0), \qquad L_0|\Delta\rangle = \Delta|\Delta\rangle. \tag{15}$$

The *Verma module* $M_{c,\Delta}$ is the representation freely generated by operators L_{-k} $(k>0)$. In other words, it is spanned on the vectors

$$L_{-k_1}L_{-k_2}\ldots L_{-k_r}|\Delta\rangle, \qquad 0 < k_1 \le k_2 \le \cdots \le k_r, \tag{16}$$

which are linearly independent. The Verma module is not necessarily irreducible. It may contain a *null vector* $|\chi\rangle$ such that

$$L_k|\chi\rangle = 0 \quad (k>0), \qquad L_0|\chi\rangle = (\Delta+N)|\chi\rangle, \qquad N>0. \tag{17}$$

In fact, it is sufficient to demand $L_1|\chi\rangle = L_2|\chi\rangle = 0$, since $L_k \propto (\mathrm{ad}_{L_1})^{k-2}L_2$ for $k>2$.

Note that if such vector exists, it is orthogonal to any other vector in the Verma module and, in particular, has a zero norm. If $\sum k_i = N$, we have

$$(|\chi\rangle, L_{-k_1}L_{-k_2}\ldots L_{-k_r}|\Delta\rangle) = \langle\Delta|L_{k_r}\ldots L_{k_2}L_{k_1}|\chi\rangle = 0.$$

If $\sum k_i \ne N$ the scalar product vanishes since it corresponds to different eigenvalues of the Hermitian operator L_0.

Null vectors generate submodules in the Verma module generated by L_{-k} ($k > 0$) and the irreducible representation is obtained by factorizing the Verma module over all such submodules:

$$V_{c,\Delta} \cong M_{c,\Delta}/\{|\chi_i\rangle \sim 0\}. \tag{18}$$

Remark. We may say that in 'reasonable' conformal field theories, which pretend to have physical applications, the space of states must consist of irreducible representations of the Virasoro algebra. In these theories null vectors are just zero in the physical space of states. Nevertheless, in some intermediate calculations or mathematical constructions 'non-reasonable' theories may be considered, where not only Verma modules, but also modules conjugate to Verma modules, or even more complicated reducible representations may appear. In what follows we will discuss 'reasonable' or 'physical' CFTs, if the opposite would not be specified explicitly.

For $N = 1$ there is the only vector $L_{-1}|\Delta\rangle$. It is a null vector if $L_1 L_{-1}|\Delta\rangle = 0$. Evidently,

$$L_1 L_{-1}|\Delta\rangle = 2L_0|\Delta\rangle = 2\Delta|\Delta\rangle.$$

Hence, $L_{-1}|\Delta\rangle$ is a null vector, if $\Delta = 0$. Since L_{-1} acts on every field as ∂, we obtain

$$\begin{aligned} \partial\Phi(z,\bar z) = 0 \text{ if } \Delta = 0; \\ \bar\partial\Phi(z,\bar z) = 0 \text{ if } \bar\Delta = 0. \end{aligned} \tag{19}$$

Good examples are $\Psi(z)$ and $\bar\Psi(\bar z)$ in the free fermion theory.

In particular, any operator of dimension $(0, 0)$ is constant. The unit operator is surely constant. If there is another operator of such dimension, it corresponds to a nontrivial vacuum on the cylinder. If all fields in a theory have dimensions $(0, 0)$ all correlation functions are constant, and the theory can be called *topological* conformal field theory.

Let us find a null vector on the level $N = 2$. A general level 2 vector has the form $(AL_{-2} + BL_{-1}^2)|\Delta\rangle$. We have

$$\begin{aligned} L_1(AL_{-2} + BL_{-1}^2)|\Delta\rangle &= (3A + 2(2\Delta + 1)B)L_{-1}|\Delta\rangle, \\ L_2(AL_{-2} + BL_{-1}^2)|\Delta\rangle &= ((4\Delta + c/2)A + 6\Delta B)|\Delta\rangle. \end{aligned}$$

The equations

$$\begin{aligned} 3A + 2(2\Delta + 1)B = 0, \\ (4\Delta + c/2)A + 6\Delta B = 0 \end{aligned}$$

have a solution if

$$\begin{vmatrix} 3 & 2(2\Delta+1) \\ 4\Delta + c/2 & 6\Delta \end{vmatrix} = -16\Delta^2 + 2(5-c)\Delta - c = 0.$$

The solutions to these equations are

$$\left.\begin{matrix} \Delta_{21} \\ \Delta_{12} \end{matrix}\right\} = \frac{1}{16}\left(5 - c \pm \sqrt{(c-1)(c-25)}\right). \tag{20}$$

Note that the dimensions Δ_{21} and Δ_{12} are real if either $c \geq 25$ or $c \leq 1$. Our basic examples of free boson ($c = 1$) and free Majorana fermion ($c = 1/2$) lay in the second region. As we shall see below the region $c < 1$ corresponds to the so called *generalized minimal conformal models*, while the region $c \geq 25$ describes the Liouville theory.

The formula looks complicated. Nevertheless, in appropriate variables it becomes simple. Parametrize the central charge by the parameter b:

$$c = 1 + 6(b + b^{-1})^2. \tag{21}$$

We have $c \geq 25$ for real b and $c \leq 1$ for purely imaginary b. Parametrize the conformal dimension by the parameter α:

$$\Delta_\alpha = \alpha(Q - \alpha). \tag{22}$$

Let

$$\alpha_{mn} = \frac{1-m}{2}b^{-1} + \frac{1-n}{2}b, \qquad \Delta_{mn} = \Delta_{\alpha_{mn}}. \tag{23}$$

In particular $\Delta_{11} = 0$ while Δ_{21}, Δ_{12} coincide with those defined in (20).

Theorem (V. Kac, 1979). *A Verma module $M_{c,\Delta}$ has a level N null vector, if and only if $\Delta = \Delta_{mn}$ with $mn = N$.*

We have seen that the existence of a level 1 null vector lead to a differential equation (19). In fact, it is a differential equation for correlation functions:

$$\frac{\partial}{\partial z}\langle\Phi_1(z_1,\bar z_1)\dots\Phi_N(z_N,\bar z_N)\Phi(z,\bar z)\rangle = 0, \quad \text{if } \Delta_\Phi = 0. \tag{24}$$

There is a natural question: if other null vectors impose on correlation functions any differential equations? To see that it is so, consider a correlation function of the form

$$\langle\Phi_1(z_1,\bar z_1)\dots\Phi_N(z_N,\bar z_N)(L_{-k}\mathcal{O})(z,\bar z)\rangle.$$

Here we will think that $\mathcal{O}(z,\bar z)$ is *any* operator (not necessarily primary), while $\Phi_1,\dots,\Phi_N$ are just primary operators of right (chiral) dimensions $\Delta_1,\dots,\Delta_N$. The action of L_{-k} on the operator is (due to the definition in the last lecture) given by

$$(L_{-k}\mathcal{O})(z,\bar z) = \oint_C \frac{dw}{2\pi\mathrm{i}}(w-z)^{1-k}T(w)\mathcal{O}(z,\bar z).$$

The contour C is a small circle around the point z. Hence, we may pull the contour to infinity. It will catch poles at $w = z_1,\dots,z_N$. We may apply the commutation relation of L_{-k} with primary fields after substitution $z \to z_i - z$. Closing around the infinity is equivalent to acting on the bra vacuum vector. Since $\langle 0|L_{-k} = 0$ we obtain

$$\begin{aligned}&\langle\Phi_1(z_1,\bar z_1)\dots\Phi_N(z_N,\bar z_N)(L_{-k}\mathcal{O})(z,\bar z)\rangle \\ &\qquad = \sum_{i=1}^N\left(\frac{(k-1)\Delta_i}{(z_i-z)^k} - \frac{1}{(z_i-z)^{k-1}}\frac{\partial}{\partial z_i}\right)\langle\Phi_1(z_1,\bar z_1)\dots\Phi_N(z_N,\bar z_N)\mathcal{O}(z,\bar z)\rangle.\end{aligned} \tag{25}$$

Evidently an mn level null vector produces a differential equation of order mn.

For example, for $\Delta = \Delta_{11} = 0$ we have

$$\sum_{i=1}^N \frac{\partial}{\partial z_i}\langle\Phi_1(z_1,\bar z_1)\dots\Phi_N(z_N,\bar z_N)\Phi(z,\bar z)\rangle = 0,$$

which is equivalent to (24) due to translation invariance.

For $\Delta_\Phi = \Delta_{21}$ or Δ_{12} we have

$$|\chi\rangle = (b^{\pm 2}L_{-1}^2 + L_{-2})|\Delta\rangle$$

and

$$\left(b^{\pm 2}\frac{\partial^2}{\partial z^2} + \sum_{i=1}^N\left(\frac{\Delta_i}{(z_i-z)^2} - \frac{1}{z_i-z}\frac{\partial}{\partial z_i}\right)\right)\langle\Phi_1(z_1,\bar z_1)\dots\Phi_N(z_N,\bar z_N)\Phi(z,\bar z)\rangle = 0. \tag{26}$$

In the next lecture we will discuss how to apply these equations to obtaining correlation functions.

Problems

1. Prove the identity (11) by using the property (prove it too):

$$\{z'',z'\} = \left(\frac{dz'}{dz}\right)^2\{z'',z'\} + \{z',z\}.$$

2. Find the third level null vectors and derive that they only exist at conformal dimensions $\Delta = \Delta_{31}, \Delta_{13}$.

3. In the free boson theory with the energy-momentum tensor

$$T(z) = -\frac{1}{4}(\partial\varphi)^2 + \frac{b+b^{-1}}{2}\partial^2\varphi$$

calculate explicitly the operators $L_{-1}V_p(z)$ and $(L_{-2}+b^{\pm 2}L_{-1}^2)V_p(z)$. Then specify them to the momenta $p = -\mathrm{i}\alpha_{mn}$ with $(m,n) = (1,1),(2,1),(1,2),(-1,-1),(-2,-1),(-1,-2)$. Show that the null vectors vanish for $m,n>0$ and remain nonvanishing for $m,n<0$.