Lecture 8
Chiral components and fermion-boson correspondence

Consider the free boson

$$\varphi(z,\bar z)=Q-2\mathrm{i}P\log z\bar z+\sum_{k\neq 0}\frac{\alpha_k z^{-k}+\bar\alpha_k\bar z^{-k}}{\mathrm{i}k}. \tag{1}$$

Its correlation function

$$\langle\varphi(z',\bar z')\varphi(z,\bar z)\rangle=2\log\frac{R^2}{(z'-z)(\bar z'-\bar z)}$$

formally splits into a some of right and left parts. Suppose that the very field splits as

$$\varphi(z,\bar z)=\varphi(z)+\bar\varphi(\bar z) \tag{2}$$

into two fields with correlation functions

$$\langle\varphi(z')\varphi(z)\rangle=2\log\frac{R}{z'-z},\qquad\langle\bar\varphi(\bar z')\bar\varphi(\bar z)\rangle=2\log\frac{R}{\bar z'-\bar z},\qquad\langle\bar\varphi(\bar z')\varphi(z)\rangle=0. \tag{3}$$

These correlation functions are multivalued functions, which reflects the nonlocal nature of these operators. The expression (1) cannot be split according to (2). To do so introduce the operators $P_\pm, Q_\pm$ with the only nonzero commutation relations

$$[P_\pm,Q_\pm]=-\mathrm{i}. \tag{4}$$

Let

$$\begin{aligned}\varphi(z)&=Q_+-2\mathrm{i}P_+\log z+\sum_{k\neq 0}\frac{\alpha_k}{\mathrm{i}k}z^{-k},\\ \bar\varphi(\bar z)&=Q_--2\mathrm{i}P_-\log\bar z+\sum_{k\neq 0}\frac{\bar\alpha_k}{\mathrm{i}k}j\bar z^{-k},\end{aligned} \tag{5}$$

The corresponding vacuum $|p_+,p_-\rangle$ is defined as

$$\begin{gathered}\alpha_k|p_+,p_-\rangle=\bar\alpha_k|p_+,p_-\rangle=0\quad(k>0),\\ P_+|p_+,p_-\rangle=p_+|p_+,p_-\rangle,\qquad P_-|p_+,p_-\rangle=p_-|p_+,p_-\rangle.\end{gathered} \tag{6}$$

The operators $\alpha_{-k},\bar\alpha_{-k}$ $(k>0)$ produce the Fock space, which we will denote $\mathcal F_{p_+}\otimes\bar{\mathcal F}_{p_-}$. The field $\varphi(z,\bar z)$ is defined according to (2) on the space

$$\mathcal H_{\text{free boson}}=\bigoplus_p\mathcal F_p\otimes\bar{\mathcal F}_p,$$

where the actions of P_+ and P_- coincide and can be identified with the action of P, and we may assume $Q=Q_++Q_-$. Indeed, $|p,p\rangle=\mathrm{e}^{\mathrm{i}p(Q_++Q_-)}|0,0\rangle$ is consistent with this assumption.

It is useful to introduce the *dual field*

$$\tilde\varphi(z,\bar z)=\varphi(z)-\bar\varphi(\bar z). \tag{7}$$

It can be easily shown that

$$\epsilon^{\mu\nu}\partial_\nu\varphi=-\partial^\mu\tilde\varphi. \tag{8}$$

where $\epsilon^{\mu\nu}$ is the 2-form ($\epsilon^{01}=-\epsilon^{10}=1$), or

$$\partial_1\varphi=-\partial_0\tilde\varphi,\qquad\partial_0\varphi=-\partial_1\tilde\varphi.$$

Hence,

$$\tilde\varphi(x)=-\int^x dy^1\,\partial_0\varphi(y)=-\int^x dy^\mu\,\epsilon_{\mu\nu}\partial^\nu\varphi(y). \tag{9}$$

For the free field theory the integral is contour independent as long as it does not cross a point, where another field is located.

The exponential operators $:\!\mathrm{e}^{\mathrm{i}p\varphi(z)}\!:$ and $:\!\mathrm{e}^{\mathrm{i}p\bar\varphi(\bar z)}\!:$ and defined in the same way as for the full field. But the corresponding correlation functions are generally multivalued:

$$\left\langle \prod_{i=1}^{\overset{\frown}{N}} :\!\mathrm{e}^{\mathrm{i}p_i\varphi(z_i)}\!: \right\rangle = \begin{cases} \prod_{i>j}^{N}(z_i-z_j)^{2p_ip_j}, & \text{if } \sum p_i = 0; \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Now consider the fields

$$V_\pm(z) = :\!\mathrm{e}^{\pm\frac{\mathrm{i}}{\sqrt2}\varphi(z)}\!:, \qquad \bar V_\pm(\bar z) = :\!\mathrm{e}^{\pm\frac{\mathrm{i}}{\sqrt2}\bar\varphi(\bar z)}\!:. \tag{11}$$

Consider the operator product

$$V_+(z')V_+(z) = (z'-z) :\!\mathrm{e}^{\frac{\mathrm{i}}{\sqrt2}(\varphi(z')+\varphi(z))}\!: = -V_+(z)V_+(z').$$

The operators V_+ (and, similarly, V_-) behave like fermions. Let us try to check this conjecture. Consider the product

$$V_-(z')V_+(z) = (z'-z)^{-1} :\!\mathrm{e}^{\frac{\mathrm{i}}{\sqrt2}(\varphi(z)-\varphi(z'))}\!: = -V_+(z)V_-(z').$$

Well. But there is a pole here. What happens at the pole? Expand the product:

$$V_-(z')V_+(z) = (z'-z)^{-1} :\!\mathrm{e}^{\frac{\mathrm{i}}{\sqrt2}(\varphi(z)-\varphi(z'))}\!: = (z'-z)^{-1} - \frac{\mathrm{i}}{\sqrt2}\partial\varphi(z) + O(z'-z).$$

First consider the singular part. In the product $V_-(z')V_+(z)$ is it assumed that the first operator is in the Euclidean future of the second: $x'^2 > x^2$. So that we may take $z' = x'^1 + \mathrm{i}0$, $z = x^1$. Then we have

$$\frac{1}{z'-z} = \frac{1}{x'^1 - x^1 + \mathrm{i}0} = \frac{1}{x'^1-x^1} - \mathrm{i}\pi\delta(x'^1-x^1).$$

Hence,

$$\begin{aligned} [V_+(x^0,x'^1), V_-(x^0,x^1)]_+ &= -2\pi\mathrm{i}\delta(x'^1-x^1), \\ [\bar V_+(x^0,x'^1), \bar V_-(x^0,x^1)]_+ &= 2\pi\mathrm{i}\delta(x'^1-x^1). \end{aligned} \tag{12}$$

We see that the operators V_+ and V_- are *not* mutually Hermitian conjugate.

If we define the product $(V_-V_+)(z)$ as the average of $V_-(z+\varepsilon\mathrm{e}^{\mathrm{i}\theta})V_+(z)$ over the value of the angle θ,

$$(V_-V_+)(z) = \int_0^{2\pi}\frac{d\theta}{2\pi}V_-(z+\varepsilon\mathrm{e}^{\mathrm{i}\theta})V_+(z) = \oint\frac{dz'}{2\pi\mathrm{i}}\frac{V_-(z')V_+(z)}{z'-z},$$

the first term will vanish, and we will obtain

$$(V_-V_+)(z) = -\frac{\mathrm{i}}{\sqrt2}\partial\varphi(z) = -\frac{\mathrm{i}}{\sqrt2}\partial\varphi(z,\bar z). \tag{13}$$

Similarly

$$(\bar V_+\bar V_-)(z) = \frac{\mathrm{i}}{\sqrt2}\bar\partial\bar\varphi(\bar z) = \frac{\mathrm{i}}{\sqrt2}\bar\partial\varphi(z,\bar z). \tag{14}$$

Introduce the notation

$$\begin{aligned} j^z_{\mathrm{top}} &= 2\bar\partial\tilde\varphi = -2\bar\partial\varphi = 2\mathrm{i}\sqrt2(\bar V_+\bar V_-), \\ j^{\bar z}_{\mathrm{top}} &= 2\partial\tilde\varphi = 2\partial\varphi = 2\mathrm{i}\sqrt2(V_-V_+). \end{aligned} \tag{15}$$

These two fields look just like components of the conserved current

$$j^\mu_{\mathrm{top}} = \epsilon^{\mu\nu}\partial_\nu\varphi = -\partial^\mu\tilde\varphi, \qquad \partial_\mu j^\mu = 0. \tag{16}$$

The corresponding charge is

$$q = \int_{-\infty}^{\infty}dx^1\,j^0 = \int_{-\infty}^{\infty}dx^1\,\partial_1\varphi = \varphi(x^0,+\infty) - \varphi(x^0,-\infty). \tag{17}$$

We will call them *topological charge* and *topological current* due to the reason, which will be clear later. Now notice that the topological charge will be conserved even in a perturbed theory, if perturbation potential prevents change of the field at infinity.

Now continue checking fermionic nature of the fields $V_\pm, \bar{V}_\pm$. Consider the product

$$V_\alpha(z')\bar{V}_\beta(z) = :\mathrm{e}^{\frac{\mathrm{i}}{\sqrt{2}}(\alpha\varphi(z')+\beta\bar{\varphi}(z))}: = \bar{V}_\beta(z)V_\alpha(z).$$

They behave like bosons despite our hopes. What to do? Let us introduce two algebraic elements η_1, η_2, which form a Clifford algebra

$$[\eta_i, \eta_j]_+ = 2\delta_{ij}. \tag{18}$$

The operators

$$\Psi_\pm(z) = \eta_1 V_\pm(z), \qquad \bar{\Psi}_\pm(\bar{z}) = \eta_2 \bar{V}_\pm(\bar{z}), \tag{19}$$

anticommute:

$$[\Psi_\alpha(z'), \Psi_\beta(z)]_+ = [\bar{\Psi}_\alpha(\bar{z}'), \bar{\Psi}_\beta(\bar{z})]_+ = [\Psi_\alpha(z'), \bar{\Psi}_\beta(\bar{z})]_+ = 0 \quad (z' \neq z, \bar{z}' \neq \bar{z}).$$

But we cannot get away with introducing the algebraic elements η_i. We should extend the space of states by tensor multiplication by a representation of the Clifford algebra. Let

$$c = \frac{\eta_1 - \mathrm{i}\eta_2}{2}, \qquad c^+ = \frac{\eta_1 + \mathrm{i}\eta_2}{2}. \tag{20}$$

Then $[c^+, c]_+ = 1$, $[c, c]_+ = [c^+, c^+]_+ = 0$. Define the states $|0\rangle_\eta, |1\rangle_\eta$ as

$$c|0\rangle_\eta = 0, \qquad |1\rangle_\eta = c^+|0\rangle_\eta. \tag{21}$$

In this basis

$$\eta_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \sigma^1, \qquad \eta_2 = \begin{pmatrix} & \mathrm{i} \\ -\mathrm{i} & \end{pmatrix} = -\sigma^2. \tag{22}$$

We may say that the total space of states is $(\mathbb{C}^2)_\eta \otimes \bigoplus_{p_+,p_-} \mathcal{F}_{p_+} \otimes \bar{\mathcal{F}}_{p_-}$.

We may suspect that $\Psi_\pm, \bar{\Psi}_\pm$ is a system of two Majorana fermions or, equivalently, a Dirac fermion. Indeed, if we consider the Dirac action

$$S_0[\psi, \bar{\psi}] = \frac{\mathrm{i}}{\pi} \int d^2x\, \bar{\psi}\gamma^\mu\partial_\mu\psi. \tag{23}$$

The difference from the Majorana action is that the field ψ is complex and that the coefficient at the action is twice as large. The complexity of ψ leads to the existence of the current

$$j^\mu = \frac{1}{\pi}\bar{\psi}\gamma^\mu\psi, \tag{24}$$

whose charge is the fermion number. It is easy to check that

$$j^z = -\frac{2}{\pi}\psi_2^+\psi_2, \qquad j^{\bar{z}} = \frac{2}{\pi}\psi_1^+\psi_1. \tag{25}$$

We may conjecture that the current j^μ and the topological current j^μ_{top} are proportional. Comparing with (15) (with $V_\pm$ substituted by $\Psi_\pm$) we see that in this case it is logical to identify $\psi_1, \psi_1^+, \psi_2, \psi_2^+$ with $\Psi_-, \Psi_+, \bar{\Psi}_+, \bar{\Psi}_-$. By comparing with the commutation

$$[\psi_i^+(x^0, x'^1), \psi_j(x^0, x^1)]_+ = \pi\delta_{ij}\delta(x'^1 - x^1), \tag{26}$$

we obtain

$$\begin{aligned} \psi_1 &= \frac{\mathrm{i}^{1/2}}{\sqrt{2}}\Psi_+, \qquad & \psi_1^+ &= \frac{\mathrm{i}^{1/2}}{\sqrt{2}}\Psi_-, \\ \psi_2 &= \frac{\mathrm{i}^{-1/2}}{\sqrt{2}}\bar{\Psi}_-, \qquad & \psi_2^+ &= \frac{\mathrm{i}^{-1/2}}{\sqrt{2}}\bar{\Psi}_+. \end{aligned} \tag{27}$$

The same factor $\mathrm{i}^{1/2}$ in ψ_1 and ψ_1^+ is related to the fact that Ψ_+ and Ψ_- are not mutually conjugate, as we have already mentioned. With the identification (27) we have

$$j^\mu_{\mathrm{top}} = 2\sqrt{2}\pi j^\mu. \tag{28}$$

And now we arrived to the most interesting point. Calculate the operator

$$\bar\psi\psi = \psi^+\gamma^0\psi = -\mathrm{i}(\psi_1^+\psi_2 - \psi_2^+\psi_1) = -\frac{\mathrm{i}}{2}(\Psi_+\bar\Psi_+ + \Psi_-\bar\Psi_-) = -\mathrm{i}\eta_1\eta_2 \,{:}\cos\frac{1}{\sqrt{2}}\varphi{:}\,.$$

This identity is crucial for establishing the correspondence between the free massive fermion and the sine-Gordon model at a special point. The action of the massive free Dirac fermion is

$$S[\psi,\bar\psi] = \frac{1}{\pi}\int d^2x\,\bar\psi(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = S_0[\psi,\bar\psi] + S_1[\psi,\bar\psi], \qquad S_1[\psi,\bar\psi] = -\frac{m}{\pi}\int d^2x\,\bar\psi\psi. \tag{29}$$

We will consider S_1 as a perturbation and define the correlation functions as

$$\langle X\rangle = \frac{\left\langle X\mathrm{e}^{\mathrm{i}S_1[\psi,\bar\psi]}\right\rangle_0}{\left\langle \mathrm{e}^{\mathrm{i}S_1[\psi,\bar\psi]}\right\rangle_0} = \langle X\rangle_0 + \mathrm{i}\left(\langle XS_1\rangle_0 - \langle X\rangle_0\langle S_1\rangle_0\right) + \cdots. \tag{30}$$

According to (29) the perturbation action can be rewritten as

$$S_1 = \frac{\mathrm{i}m}{\pi}\eta_1\eta_2\int d^2x \,{:}\cos\frac{1}{\sqrt{2}}\varphi{:}\,. \tag{31}$$

Since any correlation function must contain an even number of η_i, we may fix a vacuum $|0\rangle_\eta$ or $|1\rangle_\eta$. We have $\eta_1\eta_2|0\rangle_\eta = -\mathrm{i}|0\rangle_\eta$, $\eta_1\eta_2|1\rangle_\eta = \mathrm{i}|1\rangle_\eta$. Choose, for example, $|0\rangle_\eta$. Then

$$S_1\big|_{|0\rangle_\eta} = \frac{m}{\pi}\int d^2x \,{:}\cos\frac{1}{\sqrt{2}}\varphi{:}$$

and for the total action we get

$$S_{\mathrm{SG},\beta=\frac{1}{\sqrt{2}}}[\varphi] = \int d^2x\left(\frac{(\partial_\mu\varphi)^2}{16\pi} + \frac{m}{\pi}\cos\frac{1}{\sqrt{2}}\varphi\right). \tag{32}$$

The choice $|1\rangle_\eta$ would only change the sign at the second term, which would simply change the minimum of the potential from $\varphi = 2\sqrt{2}\pi n$ to $\varphi = \sqrt{2}\pi + 2\sqrt{2}\pi n$. Physically, it is absolutely equivalent to what we have for the $|0\rangle_\eta$ vacuum.

In classical field theory we know that the spectrum of the sine-Gordon equation

$$\partial_t^2 u - \partial_x^2 u + M^2\sin u = 0 \tag{33}$$

consists of two kinds of stable field configurations. The first kind it the *topological soliton* and *antisoliton*, which are static solutions in an appropriate inertial frame and satisfy the conditions

$$u(t,x)|_{x^1\to-\infty} = 2\pi n, \quad u(t,x)|_{x^1\to+\infty} = 2\pi(n\pm1), \quad (n\in\mathbb{Z}), \tag{34}$$

where the sign '+' corresponds to the soliton, while the sign '$-$' to the antisoliton. They are obtained by arbitrary Poincaré transformation from static solutions

$$u(t,x) = 2\pi n \pm 4\,\mathrm{arctg}\,\mathrm{e}^{Mx}. \tag{35}$$

The other type of classical excitation are the so called *breathers* which have zero topological charge, and not static in any frame. But there is a frame, where the breather does not move. In such frame it is given by

$$u(t,x) = 2\pi n + 4\,\mathrm{arctg}\,\frac{\sqrt{1-\omega^2}\cos M\omega t}{\omega\,\mathrm{ch}(M\sqrt{1-\omega^2}x)}, \qquad 0<\omega\le 1. \tag{36}$$

They form a continuous family parametrized by the variable ω. The case $\omega = 1$ corresponds to the constant solution, while the limit $\omega \to 0$ tends to a soliton–antisoliton solution. The quantity $M\omega$ is the oscillation frequency of the breather. If you calculate total energy and momentum of these configurations, it turns out that the soliton and antisoliton behave like particles of mass $8M$, while the breather behaves like a particle of mass $16M\sqrt{1-\omega^2}$. The last can be considered as a soliton–antisoliton bound state.

Here we consider a quantum sine-Gordon model. We may identify $u = \varphi/\sqrt{2}$. The topological charge q defined in (17) is quantized and is equal to $\pm 2\sqrt{2}\pi$ for the soliton and antisoliton. Due to the relation (28) we may think that in the quantum case the soliton corresponds to the fermion of the free fermion theory. Here the correspondence stops. We find no breather solutions (except the trivial one: the unbounded fermion–antifermion pair). The parameter m in the action (32) is of dimension of mass rather than of square of mass. The reason is that we consider the quantum sine-Gordon model at a very special value the Planck constant. We discuss it in more detail in the next lecture.

We may conclude that the massive free Dirac fermion is equivalent to an extension of the quantum sine-Gordon model with a special value of the coupling constant (or, equivalently, Planck constant). The equivalence is established in each order of the perturbation series in the mass m of the fermion/soliton. On one hand, this equivalence makes it possible to solve the special sine-Gordon model by reducing it to the free fermion. On the other hand, it allows one to introduce some nonlocal fields into the free fermion theory, which can be easily expressed in terms of the boson field.

Problems

1. Consider the case $m = 0$. Show that the operators $S_{\varepsilon_1\varepsilon_2}(z,\bar z) = {:}\mathrm{e}^{\frac{\mathrm{i}\varepsilon_1}{2\sqrt{2}}\varphi(z)+\frac{\mathrm{i}\varepsilon_2}{2\sqrt{2}}\bar\varphi(\bar z)}{:}$, $\varepsilon_i = \pm \equiv \pm 1$ of dimension $(1/8, 1/8)$ generate the Ramond vacuums on the cylinder. By using operator product expansion from Lecture 4 express them in terms of the operators $\sigma^{(i)}(x), \mu^{(i)}(x)$ $(i = 1, 2)$ related to the two Majorana fermions $\psi^{(i)}$ defined via the Dirac fermion $\psi(x)$ according to $\psi(x) = \frac{1}{\sqrt{2}}\left(\psi^{(1)}(x) + \mathrm{i}\psi^{(2)}(x)\right)$.

2. Consider a classical boson theory with the action

$$S[u] = \int d^2x \left(\frac{(\partial_t u)^2 - (\partial_x u)^2}{2} - V(u)\right),$$

where $V(u)$ possesses a (finite or infinite) set $\{u_i\}$ of *degenerate* minima: $V(u_i) = V_{\min}$. The set will be ordered in such a way that $u_i < u_{i+1}$. Show that any static (time-independent) solution of the equation of motion $u(x)$ 'connects' two neighboring minima: either $u(x) \to u_i$ as $x \to -\infty$ and $u(x) \to u_{i+1}$ as $x \to +\infty$ or $u(x) \to u_{i+1}$ as $x \to -\infty$ and $u(x) \to u_i$ as $x \to +\infty$.