Lecture 9
Sine-Gordon model and Thirring model

Consider the action of the sine-Gordon model in the general form

$$S_{\text{sG}}[\varphi] = \int d^2x \left(\frac{(\partial_\mu \varphi)^2}{16\pi} + 2\mu \cos \beta\varphi \right). \tag{1}$$

Here the action depends on two parameters: β and μ. The parameter β is dimensionless. We may consider it as the square root of the Planck constant: $\beta = \sqrt{\hbar}$. Indeed, let $u(x) = \beta\varphi(x)$. Then the action is

$$S_{\text{sG}} = \frac{1}{\beta^2} \int d^2x \left(\frac{(\partial_\mu u)^2}{16\pi} + 2\mu\beta^2 \cos u \right).$$

For $\beta \ll 1$ the square of mass of the lightest excitation is $m^2 = 16\pi\mu\beta^2$. If we keep it finite, the only place β enters is the prefactor, which must be $\hbar^{-1}$ in the quantum case.

We will consider the sine-Gordon theory as a perturbation of a free massless boson. The perturbation operator $:\!\cos\beta\varphi\!:$ has the scaling dimension $2\beta^2$. Hence, the dimension of the parameter μ is $2 - 2\beta^2$. Since this is the only dimensional parameter, masses of all particles in the theory must be proportional to $\mu^{1/(2-2\beta^2)}$. In the limit $\beta \to 0$ we have $m \propto \mu^{1/2}$ as expected. The behavior of the model depends of the value of β. There are three cases

1. $\beta^2 < 1$. The dimension of μ is positive, the perturbation is *relevant*. It means that the interaction decreases with decreasing scale. The short-range correlation functions behave like those in the massless free field theory. It makes the sine-Gordon theory well-defined and superrenormalizable. But the large-distance behavior strikingly differs from the short-distance one. It is described by a system of massive particles, whose mass spectrum essentially depends on the parameter β. Another interpretations comes from the fact that the sine-Gordon model describes a free massless boson (which corresponds to the dual boson $\tilde{\varphi}$) with a periodicity condition. The cosine term comes from the plasma of vortexes and massive excitation marks the Debye screening phenomenon.

2. $\beta^2 > 1$. The dimension of μ is negative and the perturbation is *irrelevant*. The interaction decreases with increasing scale, but grows indefinitely when scale decreases. The field theory in nonrenormalizable and ill-defined. The perturbation theory is effective and cannot be used beyond the first correction. In the vortex interpretation, in this region vortexes couple into neutral pairs, the Debye screening effect disappears, and the system remains massless.

3. $\beta^2 = 1$. This is the *marginal* case, where the parameter μ is dimensionless. The precise behavior at this point can be studied by means of the renormalization group method. In the case of the sine-Gordon theory this case is renormalizable, but there is no conformal field theory at short distances. The theory is massive, and its properties can be obtained by analytic continuation from the relevant perturbation region. At this point the system exhibits a Berezinsky–Kosterlitz–Thouless (BKT) transition from the vortex plasma to the vortex gas.

Let us try to find a fermion counterpart of this model. To do it, first of all, note that the topological charge

$$q = \int_{-\infty}^{\infty} dx^1\, \partial_1\varphi(x) = \varphi(+\infty) - \varphi(-\infty) \tag{2}$$

is quantized as

$$q \in \frac{2\pi}{\beta}\mathbb{Z}. \tag{3}$$

The soliton and antisoliton possess the topological charges $\pm 2\pi/\beta$. With the conjecture that the fermion is the same as quantum soliton, we get

$$j^\mu_{\text{top}} = \frac{2\pi}{\beta} j^\mu = \frac{2}{\beta}\bar{\psi}\gamma^\mu\psi. \tag{4}$$

Another conjecture is that the cosine term will correspond to the mass term of the fermion model:

$$\eta_1\eta_2 :\!\cos\beta\varphi\!: \,\propto \psi_1^+\psi_2 - \psi_2^+\psi_1. \tag{5}$$

We will search ψ_i, ψ_i^+ in the form

$$\psi_j(x) = A_j\eta_j :\mathrm{e}^{\mathrm{i}a_j\varphi(z)+\mathrm{i}b_j\bar\varphi(\bar z)}:, \qquad \psi_j^+(x) = A_j\eta_j :\mathrm{e}^{-\mathrm{i}a_j\varphi(z)-\mathrm{i}b_j\bar\varphi(\bar z)}:. \tag{6}$$

The condition that the component ψ_1 has spin $\frac{1}{2}$ and ψ_2 has spin $-\frac{1}{2}$ reads

$$a_1^2 - b_1^2 = b_2^2 - a_2^2 = \tfrac{1}{2}. \tag{7}$$

The anticommutativity of fermion fields adds

$$a_1a_2 - b_1b_2 \in \mathbb{Z}. \tag{8}$$

We demand that

$$a_1a_2 - b_1b_2 = 0 \tag{9}$$

so that the $\beta = 1/\sqrt{2}$ case was included in the construction continuously. At last, the conjecture (5) means that

$$a_1 - a_2 = b_1 - b_2 = \beta. \tag{10}$$

The equations (7), (9) and (10) can be solved and have the unique condition

$$a_1 = -b_2 = \frac{\beta}{2} + \frac{1}{4\beta}, \qquad b_1 = -a_2 = \frac{\beta}{2} - \frac{1}{4\beta}. \tag{11}$$

In other words, the definition (6) with these values can be rewritten as

$$\begin{aligned}
\psi_1(x) &= A_1\eta_1 :\mathrm{e}^{\mathrm{i}\frac{\beta}{2}\varphi(x)+\frac{\mathrm{i}}{4\beta}\tilde\varphi(x)}:, \qquad \psi_1^+(x) = A_1\eta_1 :\mathrm{e}^{-\mathrm{i}\frac{\beta}{2}\varphi(x)-\frac{\mathrm{i}}{4\beta}\tilde\varphi(x)}:,\\
\psi_2(x) &= A_2\eta_2 :\mathrm{e}^{-\mathrm{i}\frac{\beta}{2}\varphi(x)+\frac{\mathrm{i}}{4\beta}\tilde\varphi(x)}:, \quad \psi_2^+(x) = A_2\eta_2 :\mathrm{e}^{\mathrm{i}\frac{\beta}{2}\varphi(x)-\frac{\mathrm{i}}{4\beta}\tilde\varphi(x)}:,
\end{aligned} \tag{12}$$

Now study the products

$$\begin{aligned}
\psi_1^+(x')\psi_1(x) &= \frac{A_1^2}{|z'-z|^{4b_1^2}}\left(\frac{1}{z'-z} - \mathrm{i}a_1\,\partial\varphi(z) - \mathrm{i}b_1\frac{\bar z'-\bar z}{z'-z}\,\bar\partial\bar\varphi(z) + O\left(|z'-z|^2\right)\right),\\
\psi_2^+(x')\psi_2(x) &= \frac{A_2^2}{|z'-z|^{4b_1^2}}\left(\frac{1}{\bar z'-\bar z} + \mathrm{i}b_1\frac{z'-z}{\bar z'-\bar z}\,\partial\varphi(z) + \mathrm{i}a_1\,\bar\partial\bar\varphi(z) + O\left(|z'-z|^2\right)\right).
\end{aligned} \tag{13}$$

First, let us look at the first terms. The factor $(z'-z)^{-1}$ should produce a delta-function of the anticommutator $[\psi_j^+, \psi_j]_+$. But the prefactor spoils this interpretation. We could introduce a cut-off and reduce it to delta-functional form, but the result would depend on the particular method of regularization. Thus, this term is unusable for normalization of fermions. To normalize the fermions, let us use the next two terms. Let us define the product $(\psi_j^+\psi_j)(x)$ as follows

$$(\psi_j^+\psi_j)(x) = \lim_{r\to 0}\int_0^{2\pi}\frac{d\theta}{2\pi}\,\psi_j^+(z+r\mathrm{e}^{\mathrm{i}\theta}, \bar z + r\mathrm{e}^{-\mathrm{i}\theta})\psi_j(z,\bar z). \tag{14}$$

For $\psi_1^+\psi_1$ the first and third terms in (13) give zero contribution due to angular averaging, while the second term survives, but it is formally infinite. As usual, we may assume that the constant A_1 depends on the regularization cut-off r as $A_1^2 \propto r^{(\beta-1/2\beta)^2}$. Similarly, in $\psi_2^+\psi_2$ only the third term survives. Assuming (4), which reads in components

$$\psi_1^+\psi_1 = \frac{\beta}{2}\partial\varphi, \qquad \psi_2^+\psi_2 = \frac{\beta}{2}\bar\partial\bar\varphi,$$

we obtain

$$A_1^2 = -A_2^2 = \frac{\mathrm{i}}{1+1/2\beta^2}r^{4b_1^2}. \tag{15}$$

We also need the operator product expansions

$$\begin{aligned}
\psi_1(x')\psi_2^+(x) &= A_1A_2\eta_1\eta_2|z'-z|^{4a_1b_1}\left(:\mathrm{e}^{\mathrm{i}\beta\varphi(z,\bar z)}: + O(|z'-z|)\right),\\
\psi_1^+(x')\psi_2(x) &= A_1A_2\eta_1\eta_2|z'-z|^{4a_1b_1}\left(:\mathrm{e}^{-\mathrm{i}\beta\varphi(z,\bar z)}: + O(|z'-z|)\right).
\end{aligned} \tag{16}$$

It means that

$$
\begin{aligned}
\psi_1\psi_2^+ &= A_1A_2r^{\beta^2-\frac{1}{4\beta^2}}\,{:}\mathrm{e}^{\mathrm{i}\beta\varphi}{:} = \left(1+\frac{1}{2\beta^2}\right)^{-1} r^{2\beta^2-1}\,{:}\mathrm{e}^{\mathrm{i}\beta\varphi}{:}\,,\\
\psi_1^+\psi_2 &= A_1A_2r^{\beta^2-\frac{1}{4\beta^2}}\,{:}\mathrm{e}^{-\mathrm{i}\beta\varphi}{:} = \left(1+\frac{1}{2\beta^2}\right)^{-1} r^{2\beta^2-1}\,{:}\mathrm{e}^{-\mathrm{i}\beta\varphi}{:}\,.
\end{aligned}
\tag{17}
$$

We finally obtain for the cosine term

$$
\eta_1\eta_2\,{:}\cos\beta\varphi{:} = \frac{\mathrm{i}}{2}\left(1+\frac{1}{2\beta^2}\right) r^{1-2\beta^2}\,\bar\psi\psi. \tag{18}
$$

Now we want to find the Lagrangian corresponding to the fermion theory. The easiest way to do it is to reconstruct the equation of motion without the mass term. Let us find the derivative $\bar\partial\psi_1$:

$$
\bar\partial\psi_1 = \mathrm{i}b_1A_1\eta_1\,{:}\bar\partial\bar\varphi\,\mathrm{e}^{\mathrm{i}a_1\varphi(z)+\mathrm{i}b_1\bar\varphi(\bar z)}{:} = \mathrm{i}\frac{2b_1}{\beta}\lim_{r\to 0}(\psi_2^+\psi_2)(\bar z+r\mathrm{e}^{-\mathrm{i}\theta})\psi_1(z,\bar z) = \mathrm{i}\frac{2b_1}{\beta}\psi_2^+\psi_2\psi_1.
$$

After calculating in a similar way the derivative $\partial\psi_2$ we obtain the equation of motion

$$
\begin{aligned}
\bar\partial\psi_1 &= -\mathrm{i}g\psi_2^+\psi_2\psi_1,\\
\partial\psi_2 &= \mathrm{i}g\psi_1^+\psi_1\psi_2,
\end{aligned}
\tag{19}
$$

where

$$
g = \frac{1}{2\beta^2} - 1. \tag{20}
$$

In the invariant form this equation reads

$$
\mathrm{i}\gamma^\mu\partial_\mu\psi - g\gamma^\mu\psi(\bar\psi\gamma_\mu\psi) = 0. \tag{21}
$$

It corresponds to the action of the *Thirring model*:

$$
S_{\mathrm{TM}} = \frac{1}{\pi}\int d^2x\,\left(\bar\psi\mathrm{i}\gamma^\mu\partial_\mu\psi - \frac{g}{2}(\bar\psi\gamma^\mu\psi)^2\right). \tag{22}
$$

Following W. E. Thirring [1] we conclude that the Thirring model is equivalent to an extension of the free massless boson theory. Moreover, according to (18) the perturbation theory with respect to the mass term in the Thirring model is equivalent to the perturbation theory of a free boson with respect to the cosine term. Hence, the *massive Thirring model*

$$
S_{\mathrm{MTM}} = \frac{1}{\pi}\int d^2x\,\left(\bar\psi(\mathrm{i}\gamma^\mu\partial_\mu - m_0)\psi - \frac{g}{2}(\bar\psi\gamma^\mu\psi)^2\right) \tag{23}
$$

is perturbatively exactly equivalent to the (extended) sine-Gordon model with β related to g according to (20) and the *bare* mass m_0 is related to the parameter μ as

$$
\mu = \frac{m_0}{\pi}\frac{2}{1+1/2\beta^2}r^{2\beta^2-1}. \tag{24}
$$

This is the boson–fermion correspondence established by S. Coleman [2] and S. Mandelstam [3].

Formally, the bare mass m_0 has the dimension of mass, but in fact it has an anomalous dimension $1-2\beta^2$:

$$
m_0 \sim \mu r^{1-2\beta^2} \sim m_{\mathrm{phys}}^{2-2\beta^2}r^{1-2\beta^2} = m_{\mathrm{phys}}(m_{\mathrm{phys}}r)^{1-2\beta^2}. \tag{25}
$$

The exact relation (with coefficients) between the physical masses and the parameter μ was found by Al. Zamolodchikov in [4].

We may divide admissible values of the parameter g (or β) into two regions:

1. $-\frac{1}{2} \le g \le 0$, $\frac{1}{2} \ge \beta^2 \le 1$. This region corresponds to repulsion of fermions. There is no bound states and the spectrum consists of a fermion–antifermion (soliton–antisoliton) pair. At the BKT point $g < -\frac{1}{2}$ ($\beta^2 > 1$) the vacuum loses stability.

2. $g > 0$, $0 < \beta^2 < \frac{1}{2}$. This is the attraction region. There bound states of fermion and antifermion (or soliton and antisoliton), which correspond to breathers. The spectrum of breathers if discrete, and it is condensed in the classical limit $\beta^2 \to 0$ ($g \to \infty$). It is important to understand that the free fermion has no classical limit (except a formal one), and the real classical limit corresponds to a large interaction.

There is no phase transition between these two regions. At the point $g = 0$ the coupling energy of the lightest bound state simply reaches zero.

Bibliography

[1] W. E. Thirring, *Annals Phys.* **3** (1958) 91
[2] S. Coleman, *Phys. Rev.* **D11** (1975) 2088
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Problems

1. Show that in the massless Thirring model the current

$$j_3^\mu = \bar{\psi}\gamma^3\gamma^\mu\psi = \epsilon^{\mu\nu} j_\nu$$

is conserved.

2. Prove that in the massive Thirring model in the one-loop approximation of the perturbation theory in the parameter g the physical and bare mass are related as

$$m_{\rm phys} = m_0 \left(1 + g \log \frac{\Lambda}{m_0}\right),$$

where Λ is the ultraviolet momentum cut-off. Show that this formula is consistent with (25).