Lecture 10
Solving massive Thirring model by Bethe Ansatz: Bethe equation

Consider the massive Thirring model

$$S_{\rm MTM}=\frac{1}{\pi}\int d^2\xi\,\left(\bar\psi(\mathrm{i}\gamma^\mu\partial_\mu-m_0)\psi+\frac{g}{2}(\bar\psi\gamma^\mu\psi)^2\right) \tag{1}$$

on the cylinder of circumference L. Since in this lecture we will discuss the theory in the Hamiltonian form only, let us use the notation $t=\xi^0$, $x=\xi^1$. The Hamiltonian reads

$$H=\frac{1}{\pi}\int_0^L dx\,(-\mathrm{i}\psi^+\sigma^3\partial_x\psi+m_0\psi^+\sigma^2\psi+2g\psi_1^+\psi_2^+\psi_2\psi_1). \tag{2}$$

The commutation relations are

$$[\psi_\beta^+(t,x'),\psi_\alpha(t,x)]_+=\pi\delta_{\alpha\beta}\delta(x'-x). \tag{3}$$

Here we will not be interested in the vacuum energy, so that we omit the corresponding terms. There are two conserved charges in this theory: the number of particles Q and the momentum P:

$$Q=\frac{1}{\pi}\int_0^L dx\,\psi^+\psi,\qquad P=-\frac{\mathrm{i}}{\pi}\int_0^L dx\,\psi^+\partial_x\psi. \tag{4}$$

Our aim is to reduce the problem to finding a wave function in the primary quantization scheme. The problem here is that if we consider excitations over the true vacuum, the Hamiltonian contains matrix elements that relate N and $N+2$ particle wave function, where two extra particles are fermion and antifermion. To simplify the situation return to the old idea of the Dirac sea, where antifermions are interpreted as holes in the infinite sea of particles, which fill the band with negative energies. More precisely, define the *pseudovacuums* $|\Omega\rangle$ and $\langle\Omega|$:

$$\psi(x)|\Omega\rangle=0,\quad \langle\Omega|\psi^+(x)=0\quad(\forall x). \tag{5}$$

The pseudovacuum $|\Omega\rangle$ satisfies the relations

$$Q|\Omega\rangle=P|\Omega\rangle=H|\Omega\rangle=0, \tag{6}$$

if we define Q,P and H just according to (2) and (4), without any additional normal ordering. Hence, this vector is stationary and its particle number is zero. Any state of the form

$$\psi_{\alpha_N}^+(x_N)\cdots\psi_{\alpha_1}^+(x_1)|\Omega\rangle$$

is the eigenvector of Q with the eigenvalue N. It means that Q measures in this picture the number of 'particles' (we will call them *pseudoparticles*) rather the difference between numbers of particles and antiparticles, as in the usual picture. The Hamiltonian commutes with Q and does not change N. It allows us to introduce an N-particle wave function $\chi^{\alpha_1\cdots\alpha_N}(x_1,\dots,x_N)$ of the N-particle state χ_N according to

$$|\chi_N\rangle=\int d^Nx\,\chi^{\alpha_1\dots\alpha_N}(x_1,\dots,x_N)\psi_{\alpha_N}^+(x_N)\cdots\psi_{\alpha_1}^+(x_1)|\Omega\rangle. \tag{7}$$

The action of the Hamiltonian H on states can be rewritten as the action of a differential operator $\hat H_N$ on the corresponding wave function:

$$H|\chi_N\rangle=\int d^2x\,(\hat H_N\chi)^{\alpha_1\dots\alpha_N}(x_1,\dots,x_N)\psi_{\alpha_N}^+(x_N)\cdots\psi_{\alpha_1}^+(x_1)|\Omega\rangle. \tag{8}$$

It is not difficult to check that

$$\hat H_N=\sum_{k=1}^N(-\mathrm{i}\sigma_k^3\partial_{x_k}+m_0\sigma_k^2)+\pi g\sum_{k<l}^N\delta(x_k-x_l)(1-\sigma_k^3\sigma_l^3). \tag{9}$$

Consider first the case of a free fermion $g = 0$. In this case the wave function is given by the Slater determinant

$$\chi^{\alpha_1\ldots\alpha_N}_{\lambda_1\ldots\lambda_N}(x_1,\ldots,x_N) = \sum_{\sigma\in S^N}(-)^\sigma\prod_{k=1}^N \chi^{\alpha_{\sigma_k}}_{\lambda_k}(x_{\sigma_k}), \tag{10}$$

where the sum is taken over all transpositions of N numbers. The function $\chi^\alpha_\lambda(x)$ is a solution to the one-particle equation

$$(\hat{H}_1\chi_\lambda)^\alpha(x) = \epsilon_0(\lambda)\chi^\alpha_\lambda(x), \qquad \epsilon_0(\lambda) = m_0 \operatorname{ch}\lambda. \tag{11}$$

The solution to this equation is

$$\chi_\lambda(x) = \begin{pmatrix} \mathrm{e}^{\lambda/2} \\ \mathrm{i}\mathrm{e}^{-\lambda/2} \end{pmatrix} \mathrm{e}^{\mathrm{i}p_0(\lambda)x}, \qquad p_0(\lambda) = m_0 \operatorname{sh}\lambda. \tag{12}$$

The periodicity condition

$$\chi^{\alpha_1\alpha_2\ldots\alpha_N}(x_1+L,x_2,\ldots,x_N) = \mp\chi^{\alpha_1\alpha_2\ldots\alpha_N}(x_1,x_2,\ldots,x_N), \tag{13}$$

where the upper sign corresponds to the NS sector, while the lower one to the R sector, is satisfied, if

$$\mathrm{e}^{\mathrm{i}p_0(\lambda_k)L} = \mp 1, \qquad k = 1,\ldots,N. \tag{14}$$

The solution to these equation is straightforward. By taking logarithm we obtain

$$p_0(\lambda_k) = \frac{2\pi n_k}{L}, \qquad n_k \in \mathbb{Z} + \frac{\delta}{2}, \tag{15}$$

where again $\delta = 1$ in the NS sector and $\delta = 0$ in the R sector.

The solutions to these equations for λ_k maybe either real or lie on the line $\mathbb{R} + \mathrm{i}\pi$. In the first case the energies of pseudoparticles are positive, while in the second case they are negative. The total energy and momentum of the system are

$$E(\lambda_1,\ldots,\lambda_N) = \sum_{k=1}^N \epsilon_0(\lambda_k), \qquad P(\lambda_1,\ldots,\lambda_N) = \sum_{k=1}^N p_0(\lambda_k). \tag{16}$$

Due to the Pauli principle each admissible value of λ_k can be occupied by no more than one particle. Thus, if we introduce a cut-off $\Lambda \gg m_0$, the true vacuum correspond to the N-particle state with $N = \Lambda L/\pi$. An elementary excitation over this vacuum is either a pseudoparticle of positive energy or a hole, i.e. a vacancy in the negative energy band.

Since this picture is elementary since 1930, let us turn to the case $g \neq 0$. If all particles are situated at different points, the Hamiltonian does not differ from the Hamiltonian of free particles, and the wave functions are linear combinations of products of $\chi^\alpha_\lambda(x)$. Consider for simplicity the two-particle function. Due to the energy-momentum conservation the scattering of two pseudoparticles is reflectionless and we have

$$\chi^{\alpha_1\alpha_2}_{\lambda_1\lambda_2}(x_1,x_2) = \begin{cases} A_{12}\chi^{\alpha_1}_{\lambda_1}(x_1)\chi^{\alpha_2}_{\lambda_2}(x_2) - A_{21}\chi^{\alpha_1}_{\lambda_2}(x_1)\chi^{\alpha_2}_{\lambda_1}(x_2), & \text{if } x_1 < x_2; \\ A_{21}\chi^{\alpha_1}_{\lambda_1}(x_1)\chi^{\alpha_2}_{\lambda_2}(x_2) - A_{12}\chi^{\alpha_1}_{\lambda_2}(x_1)\chi^{\alpha_2}_{\lambda_1}(x_2), & \text{if } x_1 > x_2. \end{cases} \tag{17}$$

This is the only admissible antisymmetric with respect to $(\alpha_1,x_1) \leftrightarrow (\alpha_2,x_2)$ function. Up to normalization it depends on the ratio A_{12}/A_{21}. The two-particle Hamiltonian contains the interaction term

$$\pi g\delta(x_1-x_2)(1-\sigma^3_1\sigma^3_2) = \pi g\delta(x_1-x_2) \begin{array}{c} \begin{matrix} ++ & +- & -+ & -- \end{matrix} \\ \begin{pmatrix} 0 & & & \\ & 2 & & \\ & & 2 & \\ & & & 0 \end{pmatrix} \end{array} \begin{matrix} ++ \\ +- \\ -+ \\ -- \end{matrix}$$

Hence, the delta-functional term acts diagonally, and only on χ^{+-} and χ^{-+} components. Let us write the Schrödinger equation in the vicinity of the discontinuity line $x_1 = x_2$. Here we may neglect the mass term and write

$$\begin{aligned} &\mathrm{i}(\partial_1 + \partial_2)\chi^{++} = 0, \\ &\mathrm{i}(\partial_1 - \partial_2)\chi^{+-} = 2\pi g \delta(x_1 - x_2)\chi^{+-}, \\ &\mathrm{i}(\partial_2 - \partial_1)\chi^{-+} = 2\pi g \delta(x_1 - x_2)\chi^{-+}, \\ &\mathrm{i}(\partial_1 + \partial_2)\chi^{--} = 0. \end{aligned} \tag{18}$$

The first and last equations does not fix the $x_1 - x_2$ dependance of the function at al. Hence, they are consistent with any discontinuity. The second and third equations are equivalent due to the antisymmetry of the wave function. Hence, we have the only equation. If we take $x = x_1 - x_2$, we obtain

$$2\mathrm{i}\partial_x \chi^{+-} = 2\pi g \delta(x)\chi^{+-}. \tag{19}$$

This is an equation of the form

$$f'(x) = c\delta(x)f(x). \tag{20}$$

There is a difficulty in this equation. Indeed, the function $f(x)$ must have a discontinuity at $x = 0$. The discontinuity is proportional to the coefficient at $\delta(x)$. But it is $cf(0)$, which is undefined due to the very discontinuity. What to do? To solve the problem let us substitute the function $\delta(x)$ by a smooth function $\delta_\varepsilon(x)$, which is equal to zero for $|x| > \varepsilon$ and has the property $\int_{-\varepsilon}^{\varepsilon} dx\, \delta_\varepsilon(x) = 1$. We have

$$(\log f(x))' = \frac{f'(x)}{f(x)} = c\delta_\varepsilon(x).$$

Hence

$$\log \frac{f(+\varepsilon)}{f(-\varepsilon)} = \int_{-\varepsilon}^{\varepsilon} dx\, (\log f(x))' = c \int_{-\varepsilon}^{\varepsilon} dx\, \delta_\varepsilon(x) = c.$$

After taking the limit $\varepsilon \to 0$ we obtain

$$f(+0) = \mathrm{e}^c f(-0). \tag{21}$$

Hence,

$$\chi^{+-}_{\lambda_1\lambda_2}(x_2 + 0, x_2) = \mathrm{e}^{-\mathrm{i}\pi g}\chi^{+-}_{\lambda_1\lambda_2}(x_2 - 0, x_2).$$

After substituting (17) we obtain

$$\frac{A_{12}}{A_{21}} = R(\lambda_1 - \lambda_2), \qquad R(\lambda) = \mathrm{e}^{-\mathrm{i}\Phi(\lambda)} = \frac{\mathrm{ch}\,\frac{\lambda + \mathrm{i}\pi g}{2}}{\mathrm{ch}\,\frac{\lambda - \mathrm{i}\pi g}{2}}. \tag{22}$$

Below we will need the function $\Phi(\lambda)$ defined so that

$$\Phi(-\lambda) = -\Phi(\lambda)$$

and the cuts lie on the rays $(\mathrm{i}\pi g, \mathrm{i}\infty)$ and $(-\mathrm{i}\pi g, -\mathrm{i}\infty)$.

Note that all functions are periodic in g with the period 2. Since we expect that our consideration is correct for small g we will assume

$$-1 < g < 1. \tag{23}$$

This periodicity contradicts the results of the boson–fermion correspondence. We will discuss this contradiction later.

For the N-particle wave function we have the following expression

$$\chi^{\alpha_1\ldots\alpha_N}_{\lambda_1\ldots\lambda_N}(x_1, \ldots, x_N) = \sum_{\tau \in S^N} (-1)^{\sigma\tau} A_\tau \prod_{k=1}^{N} \chi^{\alpha_{\sigma_k}}_{\lambda_{\tau_k}}(x_{\sigma_k}), \quad \text{if } x_{\sigma_1} < \ldots < x_{\sigma_N}. \tag{24}$$

This form of solution is called the *coordinate Bethe Ansatz*. It turns out that the discontinuity equations for all these functions have the same form and result in

$$A_{\ldots ij \ldots} = R(\lambda_i - \lambda_j)A_{\ldots ji \ldots}. \tag{25}$$

Now impose the periodicity condition (13). It is convenient to assume that $x_1 < x_k < x_1 + L$ ($\forall k \neq 1$). By comparing the two wave functions of the form (24) we obtain the *Bethe equation*:

$$\mathrm{e}^{\mathrm{i}p_0(\lambda_k)L} = \mp \prod_{\substack{k=1\\(k\neq l)}}^{N} R(\lambda_k - \lambda_l). \tag{26}$$

Again, two signs correspond to the NS and R sectors. Since $R(0) = 1$ we may omit the additional condition for the product variable. By taking logarithm of the Bethe equation we get

$$p_0(\lambda_k)L + \sum_{l=1}^{N} \Phi(\lambda_k - \lambda_l) = 2\pi n_k, \qquad n_k \in \mathbb{Z} + \frac{\delta}{2}. \tag{27}$$

If $\{\lambda_k\}_{k=1}^{N}$ is a solution to the Bethe equation, then $\{\lambda_{\sigma_k}\}_{k=1}^{N}$ is physically equivalent solution for any transposition σ. Hence a solution can be considered as a set. Its elements are called *roots* of the Bethe equation (the terminology here differs from the standard mathematical terminology). It can be shown that for any solution to the Bethe equation all value λ_k and all values n_k are different. Moreover, there is a one-to-one correspondence between the sets $\{\lambda_k\}$ and $\{n_k\}$. It follows from the fact that the Bethe equation provides a minimum of a convex function.

In the next lecture we will discuss how to solve the Bethe equation in the thermodynamic limit $L \to \infty$, $N \to \infty$.

Problems

1. By using (8) prove (9).
2. Prove (12).
3. Derive (22).