Exact analytic solution for strong shock propagation in the expanding Friedmann universe

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Strong shock in the uniformly expanding medium

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Propagation of the strong shock in the expanding Friedman universe is investigated.

Exact analytic solution of self-similar equations is obtained, determining dependences of the radius and velocity of the shock wave on time and radius. It is obtained, that in the expanding medium the velocity of shock decreases as , $\sim t^{-1/5}$ what is slower than the shock velocity in the static uniform medium $\sim t^{-3/5}$.

The radius of the shock wave in the expanding self-gravitating medium increases $\sim t^{4/5}$, more rapidly than the shock wave radius in the uniform non-gravitating medium $\sim t^{2/5}$ .So, the shock propagates in the direction of decreasing density with larger speed, than in the static medium, due to accelerating action of the decreasing density, even in the presence of a self-gravitation.

At early stages of star and galaxy formation we may expect strong explosions at last stages of evolution of very massive primordial stars, which enrich the matter with heavy elements. Detection of heavy elements at red shifts up to z ~ 10 from GRB observations (GRB090423 at z ≈ 8.2, GRB120923A at z ≈ 8.5, GRB090429B with a photo-z ≈ 9.4) [6] make plausible this suggestion. Propagation of a strong shock in the static uniform media was investigated by many authors but the finite analytic self-similar solution was obtained by L.I. Sedov in 1946.

Here we obtain a self-similar solution for the strong shock propagating through the uniform expanding media, corresponding to the Friedman solution for the flat universe:

$$\delta = \frac{1}{6\pi G_g}, \quad \rho_1 = \delta/t^2, \qquad v_1 = 2r/3t.$$

Sedov solution for a static uniform medium

Let us first describe the self-similar solution of Sedov [3], following [1]. Hydrodynamic equations in spherically symmetric isentropic case are written as

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \quad \frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial r} + \frac{2\rho v}{r} = 0, \quad \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial r}\right)\ln\frac{p}{\rho^\gamma} = 0. \qquad (1)$$

Here v is a gas velocity in the laboratory coordinate system, ρ is the gas density, p is the gas pressure. Defining the variables with subscripts "1" and "2" corresponding to the values in front and behind the shock wave, we may write the relations on the shock, moving through the static media ($v_1 = 0$) in the form [4]

$$\begin{aligned} v_2 &= \frac{2}{\gamma+1}u_1\left(1 - \frac{c_1^2}{u_1^2}\right), \\ \rho_2 &= \frac{\gamma+1}{\gamma-1}\rho_1\left(1 + \frac{2}{\gamma-1}\frac{c_1^2}{u_1^2}\right)^{-1}, \\ p_2 &= \frac{2}{\gamma+1}\rho_1 u_1^2\left(1 - \frac{\gamma-1}{2\gamma}\frac{c_1^2}{u_1^2}\right). \end{aligned} \qquad (2)$$

Here u_1 is the shock velocity relative to the unmoving gas in front ($v_1 = 0$), γ is the adiabatic power, c_1 is the sound velocity in the static gas, $c_1^2 = \gamma\left(\frac{p_1}{\rho_1}\right)$. In the strong shock the shock velocity is much larger than the sound velocity in the undisturbed gas, $u_1 \gg c_1$, and $p_2 \gg \frac{\gamma+1}{\gamma-1}p_1$, so we have from (2)

$$v_2 = \frac{2}{\gamma+1}u_1, \;\; \rho_2 = \frac{\gamma+1}{\gamma-1}\rho_1, \;\; p_2 = \frac{2}{\gamma+1}\rho_1 u_1^2, \;\; c_2^2 = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}u_1^2.$$

There are only two parameters in the case of the strong shock: the density of the unperturbed gas ρ_1. and the total energy of the explosion E. From these parameters, together with the independent variables t, r, it is possible to construct only one non-dimensional combination $r(\rho_1/Et^2)^{1/5}$, what means that the problem has a self-similar solution. A position of the shock in the self-similar solution should correspond to fixed value of the self-similar coordinate, so that the distance of the shock to the center R may be written as

$$R = \beta\left(\frac{Et^2}{\rho_1}\right)^{1/5}, \qquad u_1 = \frac{dR}{dt} = \frac{2R}{5t} = \frac{2\beta E^{1/5}}{5\rho_1^{1/5}t^{3/5}}. \quad \xi = \frac{r}{R(t)} = \frac{r}{\beta}\left(\frac{\rho_1}{Et^2}\right)^{1/5}.$$

So, the velocity of the shock u_1, the velocity of the matter behind the shock v_2, moving through the constant density medium are decreasing with time $\sim t^{-3/5}$, and pressure behind the shock $p_2 \sim t^{-6/5}$.

Non-dimensional variables: $v = \frac{2r}{5t}V, \;\; \rho = \rho_1 G, \;\; c^2 = \frac{4r^2}{25t^2}Z.$

Self-similar equations:

$$Z\left(\frac{d\ln Z}{d\ln\xi} + \frac{d\ln G}{d\ln\xi} + 2\right) + \gamma(V-1)\frac{dV}{d\ln\xi} = \gamma V(\frac{5}{2} - V),$$

$$\frac{dV}{d\ln\xi} - (1-V)\frac{d\ln G}{d\ln\xi} = -3V,$$

$$\frac{d\ln Z}{d\ln\xi} - (\gamma-1)\frac{d\ln G}{d\ln\xi} = -\frac{5-2V}{1-V}.$$

Energy conservation integral

Using Bernoulli integral, the energy flux through the spherical surface with radius r during the time dt is written as

$$dq_{-} = dt\, 4\pi r^2 \rho v (w + \frac{v^2}{2})$$

The gain of the energy dq+ is due to increase of the volume of this sphere during the time dt by the value $(dt\, v_n 4\pi r^2)$, containing the energy

$$dq_{+} = dt\, 4\pi r^2 \rho v_n (\varepsilon + \frac{v^2}{2}), \quad w = \varepsilon + \frac{p}{\rho}.$$

$$v_n = \frac{2r}{5t} = u_1 \frac{r}{R}.$$

$$q_- = q_+ \qquad v\left(w + \frac{v^2}{2}\right) = v_n\left(\varepsilon + \frac{v^2}{2}\right). \qquad \varepsilon = \frac{c^2}{\gamma(\gamma-1)}, \quad w = \frac{c^2}{\gamma-1},$$

$$v\left(\frac{c^2}{\gamma-1} + \frac{v^2}{2}\right) = v_n\left(\frac{c^2}{\gamma(\gamma-1)} + \frac{v^2}{2}\right). \qquad \frac{c^2}{\gamma-1}\left(V - \frac{1}{\gamma}\right) = \frac{v^2}{2}(1-V)$$

The relation between self-similar variables, following from the energy conservation:

$$Z = \frac{\gamma(\gamma-1)(1-V)}{2(\gamma V - 1)} V^2.$$

Identity on the shock boundary

Excluding density ***G*** from self-similar equations we obtain:

$$(\gamma - 1)\frac{dV}{d\ln\xi} - (1 - V)\frac{d\ln Z}{d\ln\xi} = 5 - 3\gamma V + V.$$

From the energy integral it follows:

$$\frac{d\ln Z}{d\ln\xi} = -\frac{1}{1-V}\,\frac{dV}{d\ln\xi} + 2\frac{1}{V}\,\frac{dV}{d\ln\xi} - \frac{\gamma}{\gamma V - 1}\,\frac{dV}{d\ln\xi}.$$

The equation for ***V***

$$\left(\gamma + 1 - \frac{2}{V} + \frac{\gamma - 1}{\gamma V - 1}\right)\frac{dV}{d\ln\xi} = 5 - 3\gamma V + V.$$

which has the analytical solution (Sedov, 1946):

$$\left(\frac{\gamma+1}{2}V\right)^{-2}\left[\frac{\gamma+1}{\gamma-1}(\gamma V-1)\right]^{\nu_1}\left[\frac{\gamma+1}{7-\gamma}(5-3\gamma V+V)\right]^{\nu_2}=\xi^5,$$

$$\nu_1=5\frac{\gamma-1}{2\gamma+1},\quad \nu_2=-\frac{13\gamma^2-7\gamma+12}{(3\gamma-1)(2\gamma+1)}.$$

The solution satisfies boundary conditions on the shock for self-similar variables:

$$V(1)=\frac{2}{\gamma+1},\ \ G(1)=\frac{\gamma+1}{\gamma-1},\ \ Z(1)=\frac{2\gamma(\gamma-1)}{(\gamma+1)^2}.$$

The equation for *G(V)*

$$(1-V)\frac{d\ln G}{dV}=\frac{2(1+2V)+3\frac{1-V}{\gamma V-1}}{5-3\gamma V+V}.$$

has the solution (Sedov, 1946):

$$G(V)=\frac{\gamma+1}{\gamma-1}\left[\frac{\gamma+1}{\gamma-1}(1-V)\right]^{\sigma_1}\left[\frac{\gamma+1}{\gamma-1}(\gamma V-1)\right]^{\sigma_2}$$
$$\times\left[\frac{\gamma+1}{7-\gamma}(5-3\gamma V+V)\right]^{\sigma_3},$$

$$\sigma_1=-\frac{2}{2-\gamma},\ \ \sigma_2=\frac{3}{2\gamma+1},\ \ \sigma_3=\frac{13\gamma^2-7\gamma+12}{(2-\gamma)(3\gamma-1)(2\gamma+1)}.$$

The constant β in the definition of the non-dimensional radius ξ

is defined by the known energy of the explosion ***E***

$$E = \int_0^{R(t)} \rho \left[\frac{v^2}{2} + \frac{c^2}{\gamma(\gamma-1)} \right] 4\pi r^2 dr.$$

$$1 = \beta^5 \frac{16\pi}{25} \int_0^1 G \left[\frac{V^2}{2} + \frac{Z}{\gamma(\gamma-1)} \right] \xi^4 d\xi$$

for $\gamma = \frac{7}{5}$ $\beta = 1.033.$

$$\beta(\gamma) \sim 1.$$

Strong shock in a uniform expanding medium

$$\rho_1 = \delta / t^2, \qquad v_1 = 2r/3t. \qquad \delta = \frac{1}{6\pi G_g}, \quad \rho_1 = \frac{1}{6\pi G_g t^2}, \quad \frac{G_g m}{r^2} = \frac{2}{9}\frac{r}{t^2}.$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{G_g m}{r^2}, \quad \frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial r} + \frac{2\rho v}{r} = 0,$$

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial r}\right)\ln\frac{p}{\rho^\gamma} = 0, \quad \frac{\partial m}{\partial r} = 4\pi\rho r^2.$$

$$v_1 \ll c_{\text{light}}, \quad c \ll c_{\text{light}}.$$

c is the sound velocity

Conditions on the shock in expanding medium, ***u*** is a shock velocity:

$$v_2 = \frac{2}{\gamma+1}u + \frac{\gamma-1}{\gamma+1}v_1^{sh}, \;\; \rho_2 = \frac{\gamma+1}{\gamma-1}\rho_1,$$

$$p_2 = \frac{2}{\gamma+1}\rho_1(u - v_1^{sh})^2, \;\; c_2^2 = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}(u - v_1^{sh})^2.$$

energy E, the number of parameters is the same as in the previous static medium $(\delta, \ E)$, therefore we look for a self-similar solution for the case of a strong shock motion. The non-dimensional combination in the case of a uniformly expanding medium is written as $r(\delta/Et^4)^{1/5}$. A position of the shock in the self-similar solution should correspond to the fixed value of the self-similar coordinate, so that the distance of the shock to the center R may be written as

$$R = \beta \left(\frac{Et^4}{\delta}\right)^{1/5}, \qquad u = \frac{dR}{dt} = \frac{4R}{5t} = \frac{4\beta E^{1/5}}{5\delta^{1/5}t^{1/5}}.$$

Non-dimensional variables: $v = \frac{4r}{5t}V,\ \rho = \frac{\delta}{t^2}G,\ c^2 = \frac{16r^2}{25t^2}Z,\ m = \frac{4\pi}{3}\rho_1 r^3 M = \frac{4\pi}{3}\frac{r^3}{t^2}\delta M$

Self-similar equations:

$$Z\left(\frac{d\ln Z}{d\ln\xi} + \frac{d\ln G}{d\ln\xi} + 2\right) + \gamma(V-1)\frac{dV}{d\ln\xi} = \gamma V(\frac{5}{4} - V) - \frac{25}{72}\gamma M,$$

$$\frac{dV}{d\ln\xi} - (1-V)\frac{d\ln G}{d\ln\xi} = -3V + \frac{5}{2},$$

$$\frac{d\ln Z}{d\ln\xi} - (\gamma-1)\frac{d\ln G}{d\ln\xi} = -\frac{5 - 2V - \frac{5}{2}\gamma}{1-V},$$

$$\xi\frac{dM}{d\xi} = 3(G-M).$$

Self-similar variable:

$$\xi = \frac{r}{R(t)} = \frac{r}{\beta}\left(\frac{\delta}{Et^4}\right)^{1/5}$$

Conditions on the shock

$$V(1) = \frac{5\gamma + 7}{6(\gamma + 1)}, \quad G(1) = \frac{\gamma + 1}{\gamma - 1}, \quad Z(1) = \frac{\gamma(\gamma - 1)}{18(\gamma + 1)^2}, \quad M(1) = 1.$$

The first integral of the problem

Writing the energy conservation in the co-moving frame, we obtain

$$(v - v_1^{sh}\frac{r}{R})\left(\frac{c^2}{\gamma-1} + \frac{(v - v_1^{sh}\frac{r}{R})^2}{2}\right) = (v_n - v_1^{sh}\frac{r}{R})(\frac{c^2}{\gamma(\gamma-1)} + \frac{(v - v_1^{sh}\frac{r}{R})^2}{2}).$$

In non-dimensional variables:

$$\left(\frac{4r}{5t}V - \frac{2r}{3t}\right)\left[\frac{Z}{\gamma-1}\frac{16r^2}{25t^2} + \frac{1}{2}\left(\frac{4r}{5t}V - \frac{2r}{3t}\right)^2\right] =$$

$$\left(\frac{4r}{5t} - \frac{2r}{3t}\right)\left[\frac{Z}{\gamma(\gamma-1)}\frac{16r^2}{25t^2} + \frac{1}{2}\left(\frac{4r}{5t}V - \frac{2r}{3t}\right)^2\right].$$

$$v_n = \frac{4r}{5t}, \quad v_1^{sh} = \frac{2r}{3t}, \quad v = \frac{4r}{5t}V$$

It reduces to:

$$\left(V-\frac{5}{6}\right)\left[\frac{Z}{\gamma-1}+\left(V-\frac{5}{6}\right)^2\right]=\left(1-\frac{5}{6}\right)\left[\frac{Z}{\gamma(\gamma-1)}+\left(V-\frac{5}{6}\right)^2\right]$$

The relation between self-similar variables, following from the energy conservation in the co-moving frame:

$$Z=\frac{(\gamma-1)(1-V)\left(V-\frac{5}{6}\right)^2}{2\left(V-\frac{5}{6}-\frac{1}{6\gamma}\right)}.$$

Identity on the shock boundary

Excluding density ***G*** from self-similar equations we obtain:

$$(\gamma - 1)\frac{dV}{d\ln\xi} - (1 - V)\frac{d\ln Z}{d\ln\xi} = \frac{5}{2} - 3\gamma V + V.$$

From the energy integral it follows:

$$\frac{d\ln Z}{d\ln\xi} = -\frac{1}{1-V}\frac{dV}{d\ln\xi} + \frac{2}{V - \frac{5}{6}}\frac{dV}{d\ln\xi} - \frac{\gamma}{\gamma V - \frac{5\gamma}{6} - \frac{1}{6}}\frac{dV}{d\ln\xi}.$$

The equation for ***V***

$$\left(\gamma + 1 - \frac{2}{6V - 5} + \frac{\gamma - 1}{6\gamma V - 5\gamma - 1}\right)\frac{dV}{d\ln\xi} = \frac{5}{2} - 3\gamma V + V.$$

has the analytical solution similar to Sedov case:

$$\left[(\gamma+1)(3V-\frac{5}{2})\right]^{\mu_1}\left[\frac{\gamma+1}{\gamma-1}(6\gamma V-5\gamma-1)\right]^{\mu_2}\left[6(\gamma+1)\frac{3\gamma V-V-\frac{5}{2}}{15\gamma^2+\gamma-22}\right]^{\mu_3}=\xi,$$

$$\mu_1=\frac{2}{15\gamma-20},\quad \mu_2=\frac{\gamma-1}{17\gamma-15\gamma^2+1},$$

$$\mu_3=-\frac{\gamma+1}{3\gamma-1}-\frac{\gamma-1}{17\gamma-15\gamma^2+1}+\frac{2}{20-15\gamma}.$$

The solution satisfies boundary conditions on the shock for self-similar variables:

$$V(1)=\frac{5\gamma+7}{6(\gamma+1)},\quad G(1)=\frac{\gamma+1}{\gamma-1},\quad Z(1)=\frac{\gamma(\gamma-1)}{18(\gamma+1)^2}.$$

The equation for *G(V)*

$$1-(1-V)\frac{d\ln G}{dV}=-(3V-\frac{5}{2})\frac{\gamma+1-\frac{2}{6V-5}+\frac{\gamma-1}{6\gamma V-5\gamma-1}}{\frac{5}{2}-V(3\gamma-1)}.$$

has the following solution:

$$G(V)=\frac{\gamma+1}{\gamma-1}\left[6\frac{(\gamma+1)(1-V)}{\gamma-1}\right]^{\kappa_1}\left[\frac{\gamma+1}{\gamma-1}(6\gamma V-5\gamma-1)\right]^{\kappa_2}$$
$$\times\left(\frac{3(\gamma+1)}{15\gamma^2+\gamma-22}[(6\gamma-2)V-5)]\right)^{\kappa_3}.$$

Where

$$\kappa_1 = \frac{7}{3\gamma - 1} - \frac{2}{6\gamma - 7} + \frac{(15\gamma - 20)(\gamma - 1)}{(6\gamma - 7)(15\gamma^2 - 17\gamma - 1)}$$

$$- \frac{3\gamma(15\gamma - 20)}{(3\gamma - 1)(15\gamma^2 - 17\gamma - 1)} - \frac{15\gamma - 20}{3\gamma - 1}\frac{\gamma + 1}{6\gamma - 7},$$

$$\kappa_2 = -\frac{3}{3\gamma - 1} + \frac{3\gamma(15\gamma - 20)}{(3\gamma - 1)(15\gamma^2 - 17\gamma - 1)}.$$

$$\kappa_3 = \frac{2}{6\gamma - 7} - \frac{(15\gamma - 20)(\gamma - 1)}{(6\gamma - 7)(15\gamma^2 - 17\gamma - 1)} + \frac{15\gamma - 20}{3\gamma - 1}\frac{\gamma + 1}{6\gamma - 7}$$

The constant β in the definition of the non-dimensional radius ξ

is defined by the known energy of the explosion ***E***

$$E = \int_0^{R(t)} \rho \left[\frac{\left(v - \frac{2r}{3t}\right)^2}{2} + \frac{c^2}{\gamma(\gamma-1)} \right] 4\pi r^2 dr.$$

$$1 = \beta^5 \frac{64\pi}{25} \int_0^1 G \left[\frac{\left(V - \frac{5}{6}\right)^2}{2} + \frac{Z}{\gamma(\gamma-1)} \right] \xi^4 d\xi$$

Expectation: $\beta(\gamma) \sim 1.$

Physically relevant self-similar solutions exist only for limited interval of (γ)

Conclusions

It follows from the self-similar solution, that in the expanding medium the velocity of shock from (41) decreases as $\sim t^{-1/5}$, what is much slower than the shock velocity in the static uniform medium $\sim t^{-3/5}$, according to Sedov solution (5). Correspondingly the radius of the shock wave in the expanding self-gravitating medium increases $\sim t^{4/5}$, more rapidly that the shock wave radius in the uniform non-gravitating medium $\sim t^{2/5}$. It means, that the shock propagates in the direction of decreasing density with larger speed, than in the static medium, due to accelerating action of the decreasing density, even in the presence of a self-gravitation.

Solution may be applied also for explosions inside the expanding media – Supernovae in the stellar wind.