Zeros of polynomials and solvable nonlinear evolution equations

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Abstract

Recent findings concerning the relations between the coefficients and the zeros of time-dependent monic polynomials will be reported. They underline a differential algorithm to compute all the zeros of a generic polynomial and allow the identification of novel classes of solvable nonlinear evolution equations, including many-body problems and nonlinear evolution partial differential equations, as well as endless hierarchies of such solvable models. Part of this work has been done with **Oksana Bihun**, with **Mario Bruschi** and with **Francois Leyvraz**.

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Coefficients and zeros of monic polynomials

$$p_N(z;\ \vec{y};\tilde{x}) = z^N + \sum_{m=1}^{N} (y_m\, z^{N-m})$$

$$p_N(z;\ \vec{y};\tilde{x}) = \prod_{n=1}^{N} (z - x_n)$$

$\vec{y}$ is an *N*-vector: its *N* components y_m are the *N* *coefficients* of the polynomial $p_N(z;\ \vec{y};\tilde{x})$ of degree *N* in the independent (*complex*) variable *z.*

$\tilde{x}$ is an ***unordered*** set of *N* numbers x_n, which are the *N* *zeros* of the polynomial $p_N(z;\ \vec{y};\tilde{x})$. We denote as $\vec{x}_{[\mu]}$ the *N*-vector, the *N* components $x_{[\mu],n}$ of which are given by the specific permutation--- labeled by the integer index μ in the range $1 \leq \mu \leq N$!---of the *N* numbers x_n . Hence, to any given $\tilde{x}$, there generally correspond *N*! *different* *N*-vectors $\vec{x}_{[\mu]}$. We will come back to this notion below.

Time-dependent monic polynomials

$$\psi_N\left(\zeta;\ \vec{\gamma}(t); \tilde{\vec{\xi}}(t)\right) = \zeta^N + \sum_{m=1}^{N} [\gamma_m(t)\, \zeta^{N-m}]$$

$$\psi_N\left(\zeta;\ \vec{\gamma}(t); \tilde{\vec{\xi}}(t)\right) = \prod_{n=1}^{N} [\zeta - \xi_n(t)]$$

$\vec{\gamma}(t)$ is a time-dependent *N*-vector: its *N* components $\gamma_m(t)$ are the *N coefficients* of the time-dependent polynomial $\psi_N\left(\zeta;\ \vec{\gamma}(t); \tilde{\vec{\xi}}(t)\right)$ of degree *N* in the independent (generally *complex*) variable ζ. $\tilde{\vec{\xi}}(t)$ is an ***unordered*** set of *N* numbers $\xi_n(t)$, which are the *N zeros* of the polynomial $\psi_N\left(\zeta;\ \vec{\gamma}(t); \tilde{\vec{\xi}}(t)\right)$ (but they may get *ordered* via their dependence on time if this dependence is *continuous*).

Two very useful formulas [1]

$$\dot{\xi}_n(t) = -\left\{ \prod_{j=1, j\neq n}^{N} \left[\xi_n(t) - \xi_j(t)\right] \right\}^{-1} \sum_{m=1}^{N} \{\dot{\gamma}_m(t)\, [\xi_n(t)]^{N-m}\}\,,$$

$$\ddot{\xi}_n(t) = \sum_{j=1, j\neq n}^{N} \left[\frac{2\, \dot{\xi}_n(t)\, \dot{\xi}_j(t)}{\xi_n(t) - \xi_j(t)} \right]$$

$$-\left\{ \prod_{j=1, j\neq n}^{N} \left[\xi_n(t) - \xi_j(t)\right] \right\}^{-1} \sum_{m=1}^{N} \left\{\ddot{\gamma}_m(t)\, \left[\xi_n(t)\right]^{N-m}\right\}\,,$$

$$\gamma_m(t) = (-1)^m\, \sigma_m\left[\tilde{\xi}(t)\right]\,.$$

The second-derivative formula is particularly useful and is referred below as (***).

Extension to higher derivatives

These formulas have been extended to the third and fourth derivatives in [3], to derivatives of *arbitrary* order in [7] and to the case of *discrete time* in [15]. These results are not reported nor discussed in this talk, but they are instrumental to demonstrate the *solvable* character of some of the dynamical systems reported below and all the development described in the 18 References (see page 2).

Nonlinear differential algorithm to compute all the zeros of a generic polynomial [13,14]

$$P_N(z;\ \vec{c};\tilde{x}) = z^N + \sum_{m=1}^{N}(c_m\, z^{N-m}) = \prod_{n=1}^{N}(z - x_n)$$

Now introduce the *t*-dependent polynomial

$$p_N\big(z;\ \vec{\gamma}(t);\tilde{y}(t)\big) = z^N + \sum_{m=1}^{N}[\gamma_m\,(t)\, z^{N-m}]$$

$$= \prod_{n=1}^{N}[z - y_n(t)]$$

There holds then the following

Proposition. Consider the following system of *N* nonlinear first-order differential equations satisfied by the *N* zeros $y_n(t)$ of this polynomial:

$$\dot{y}_n(t) = -\left\{\prod_{s=1, s\neq n}^{N} [y_n(t) - y_s(t)]^{-1}\right\} \cdot$$

$$\cdot \sum_{m=1}^{N} \left\{\dot{f}_m(t)\,[c_m - \gamma_m(0)]\,[y_n(t)]^{N-m}\right\} ,$$

$$f_m(T) - f_m(0) = 1,$$

$$\gamma_m(0) = (-1)^m\,\sigma_m(\tilde{y}\,(0)),\;\; m = 1, \cdots, N\,;$$

$$\sigma_m(\tilde{y}) = \sum_{1\leq s_1<s_2<\cdots<s_m\leq N} \left(y_{s_1}\,y_{s_2}\cdots y_{s_m}\right)\;.$$

Then: $$x_n = y_n(T)\quad .$$

It is thus seen that the zeros x_n of the polynomial $P_N(z;\ \vec{c};\tilde{x})$ can be computed---once the *N* coefficients c_m of this polynomial have been assigned---via the following procedure. *Step one*: choose (arbitrarily!) *N* complex numbers $y_n(0)$. *Step two*: compute the *N* quantities $\gamma_m(0)$. *Step three*: integrate (numerically) the above system of differential equations from *t=0* to *t=T*, starting from the *N* initial data $y_n(0)$, getting thereby the *N* values $y_n(T)$, which give the sought result, $x_n = y_n(T)$.

[Subcase: $f_m(t) = t, \dot{f}_m(t) = 1,\ \ T = 1$.]

New solvable dynamical systems

Many new *solvable* models---including *isochronous* ones---can be manufactured by taking advantage of the formula (***):

$$\ddot{\xi}_n(t) = \sum_{j=1, j\neq n}^{N} \left[\frac{2\,\dot{\xi}_n(t)\,\dot{\xi}_j(t)}{\xi_n(t)-\xi_j(t)}\right]$$

$$-\left\{\prod_{j=1, j\neq n}^{N}\left[\xi_n(t)-\xi_j(t)\right]\right\}^{-1} \sum_{m=1}^{N}\left\{\ddot{\gamma}_m(t)\,\left[\xi_n(t)\right]^{N-m}\right\},$$

$$\gamma_m(t) = (-1)^m\,\sigma_m\left[\tilde{\xi}(t)\right],$$

$$\dot{\gamma}_m(t) = (-1)^m\,\sum_{n=1}^{N}\left\{\sigma_{n,m}\left[\tilde{\xi}(t)\right]\dot{\xi}_n(t)\right\}.$$

Indeed, assume that a "solvable" dynamical system of Newtonian type ("acceleration equal force") reads

$$\ddot{\vec{\gamma}}(t) = \vec{f}\left[\vec{\gamma}(t); \dot{\vec{\gamma}}(t); t\right]$$

or equivalently

$$\ddot{\gamma}_m(t) = f_m\left[\vec{\gamma}(t); \dot{\vec{\gamma}}(t); t\right], \quad m = 1, \dots, N \, .$$

Then insert these equations of motion in the right-hand side of (***). In this manner you obtain a new "solvable" dynamical system of Newtonian type. Several new examples are exhibited in references [1-11, 18], see above. Here I display only one *well-known* example and two of those *new* examples.

The goldfish model

$$\ddot{\gamma}_m(t) = \boldsymbol{i}\,\omega\,\dot{\gamma}_m\,(t)\ ,$$

$$\gamma_m\,(t) = \gamma_m\,(0)\ + \frac{\boldsymbol{i}\,\dot{\gamma}_m(0)\,[1-\exp(\boldsymbol{i}\omega t)]}{\omega}\ ,$$

$$\ddot{\xi}_n(t) = \boldsymbol{i}\,\omega\,\dot{\xi}_n(t) + \sum_{j=1,j\neq n}^{N}\left[\frac{2\,\dot{\xi}_n(t)\,\dot{\xi}_j(t)}{\xi_n(t)-\xi_j(t)}\right]\,.$$

Solution: the *N* coordinates $\xi_n(t)$ are the *N* roots of

$$\sum_{n=1}^{N}\left[\frac{\dot{\xi}_n(0)}{\xi-\xi_n(0)}\right] = \frac{\boldsymbol{i}\,\omega}{\exp(\boldsymbol{i}\,\omega\,t)-1}\ .$$

Isochronous with period $T = 2\pi/\omega$---or a, generally small (much less than *N*!), *integer* multiple of *T*.

The “goldfish-CM” model [2]

$$\ddot{\gamma}_m(t) = -\omega^2 \gamma_m(t) + 2\, g^2 \sum_{j=1, j\neq m}^{N} \left[\gamma_m(t) - \gamma_j(t)\right]^{-3} ;$$

$$\ddot{\xi}_n(t) = \sum_{j=1, j\neq n}^{N} \left[\frac{2\, \dot{\xi}_n(t)\, \dot{\xi}_j(t)}{\xi_n(t) - \xi_j(t)}\right]$$

$$-\left\{\prod_{j=1, j\neq n}^{N} \left[\xi_n(t) - \xi_j(t)\right]\right\}^{-1} \sum_{m=1}^{N} \left\{\ddot{\gamma}_m(t) \left[\xi_n(t)\right]^{N-m}\right\} ,$$

$$\gamma_m(t) = (-1)^m \sigma_m\left[\tilde{\xi}(t)\right] .$$

Isochronous with period $T = 2\pi/\omega$---or a, generally small (much less than *N*!), *integer* multiple of *T*.

The “goldfish-CM” model equations of motion [2]

$$\ddot{\xi}_n(t) = \sum_{j=1, j\neq n}^{N} \left[\frac{2\,\dot{\xi}_n(t)\,\dot{\xi}_j(t)}{\xi_n(t) - \xi_j(t)}\right] - \left\{\prod_{j=1, j\neq n}^{N} \left[\xi_n(t) - \xi_j(t)\right]\right\}^{-1} \cdot$$

$$\cdot \sum_{m=1}^{N} \left\{\left\{-\omega^2\,\gamma_m(t) + 2\,g^2 \sum_{j=1, j\neq m}^{N} \left[\gamma_m(t) - \gamma_j(t)\right]^{-3}\right\} \left[\xi_n(t)\right]^{N-m}\right\},$$

$$\gamma_m(t) = (-1)^m\,\sigma_m\left[\tilde{\xi}(t)\right],$$

$$\sigma_m(\tilde{\xi}) = \sum_{1\leq s_1 < s_2 < \cdots < s_m \leq N} \left(\xi_{s_1}\,\xi_{s_2} \cdots \xi_{s_m}\right).$$

An *integrable Hamiltonian N*-body problem in the plane featuring *N arbitrary* functions [11]

$$\ddot{\xi}_n(t) = \sum_{j=1, j\neq n}^{N} \left[\frac{2\,\dot{\xi}_n(t)\,\dot{\xi}_j(t)}{\xi_n(t)-\xi_j(t)}\right]$$

$$-\left\{\prod_{j=1, j\neq n}^{N} \left[\xi_n(t)-\xi_j(t)\right]\right\}^{-1} \sum_{m=1}^{N} \left\{g_m\big(\gamma_m(t)\big)\left[\xi_n(t)\right]^{N-m}\right\},$$

$$\gamma_m(t) = (-1)^m\,\sigma_m\left[\tilde{\xi}(t)\right],$$

$$\sigma_m(\tilde{\xi}) = \sum_{1\leq s_1<s_2<\cdots<s_m\leq N} \left(\xi_{s_1}\xi_{s_2}\cdots\xi_{s_m}\right).$$

This finding is an obvious consequence of the fact that the system of ***decoupled*** ODEs

$$\ddot{\gamma}_m(t) = g_m\big(\gamma_m(t)\big)$$

is clearly both ***Hamiltonian*** and ***integrable***; and of course both these properties are inherited by the above system of nonlinear coupled second-order ODEs satisfied by the "particle coordinates" $\xi_n(t)$ because the transformation from the *N* coordinates $\gamma_m(t)$ to the *N* coordinates $\xi_n(t)$ is clearly a ***canonical*** transformation in the Hamiltonian context.

These examples are obtained directly from the formula (***). But the process can be iterated over and over again.

This motivated the idea to introduce and investigate **the generations of monic polynomials obtained by replacing the *coefficients* of the polynomials of the next generation with the *zeros* of a polynomial of the previous generation. [4]**

The *seed* polynomial

$$p_N^{(0)}\left(z;\ \vec{y}^{(0)}; \tilde{x}^{(0)}\right) = z^N + \sum_{m=1}^{N} \left[y_m^{(0)}\ z^{N-m}\right],$$

$$p_N^{(0)}\left(z;\ \vec{y}^{(0)}; \tilde{x}^{(0)}\right) = \prod_{n=1}^{N} \left[z - x_n^{(0)}\right].$$

The first generation of *N*! monic polynomials

$$p_N^{(\mu_1;1)}\left(z; \vec{y}_{[\mu_1]}^{(1)}\ ; \tilde{x}_{[\mu_1]}^{(1)}\right) = z^N + \sum_{m=1}^{N} \left[y_{[\mu_1],m}^{(1)}\ z^{N-m}\right],$$

$$p_N^{(\mu_1;1)}\left(z; \vec{y}_{[\mu_1]}^{(1)}\ ; \tilde{x}_{[\mu_1]}^{(1)}\right) = \prod_{n=1}^{N} \left[z - x_{[\mu_1],n}^{(1)}\right],$$

$$\vec{y}_{[\mu_1]}^{(1)} = \vec{x}_{[\mu_1]}^{(0)}\ ; \quad y_{[\mu_1],m}^{(1)} = x_{[\mu_1],m}^{(0)}\ .$$

The *integer* index $\boldsymbol{\mu_1}$ taking values in the range $1 \leq \boldsymbol{\mu_1} \leq N!$ labels the permutations of the *a priori* unordered set $\tilde{x}^{(0)}$.

The second generation of $(N!)^2$ monic polynomials

$$p_N^{(\vec{\mu}^{(2)};2)}\left(z;\vec{y}^{(2)}_{[\vec{\mu}^{(2)}]}\,;\tilde{x}^{(2)}_{[\vec{\mu}^{(2)}]}\right)=z^N+\sum_{m=1}^{N}\left[y^{(2)}_{[\vec{\mu}^{(2)}],m}\,z^{N-m}\right],$$

$$p_N^{(\vec{\mu}^{(2)};2)}\left(z;\vec{y}^{(2)}_{[\vec{\mu}^{(2)}]}\,;\tilde{x}^{(2)}_{[\vec{\mu}^{(2)}]}\right)=\prod_{n=1}^{N}\left[z-x^{(2)}_{[\vec{\mu}^{(2)}],n}\right],$$

$$\vec{y}^{(2)}_{[\vec{\mu}^{(2)}]}=\vec{x}^{(1)}_{[\vec{\mu}^{(2)}]}=\vec{x}^{(1)}_{[\mu_1],[\mu_2]};\quad y^{(2)}_{[\vec{\mu}^{(2)}],m}=x^{(1)}_{[\vec{\mu}^{(2)}],m}\,.$$

The 2-vector $\vec{\mu}^{(2)}=(\mu_1,\mu_2)$ has *integer* components μ_1, μ_2 taking values in the range $1\leq\mu_1,\ \mu_2\leq N!$. The label μ_2 identifies the permutation of the *a priori unordered* set $\tilde{x}^{(1)}_{[\mu_1]}$ the components of which are the *N* zeros of the polynomial of the previous generation with index μ_1, i. e. the *N* zeros of $p_N^{(\mu_1;1)}\left(z;\vec{y}^{(1)}_{[\mu_1]}\,;\tilde{x}^{(1)}_{[\mu_1]}\right)$.

The *k*-th generation of (*N*!)^{k} monic polynomials

$$p_N^{(\vec{\mu}^{(k)};k)}\left(z;\vec{y}^{(k)}_{[\vec{\mu}^{(k)}]};\tilde{x}^{(k)}_{[\vec{\mu}^{(k)}]}\right)=z^N+\sum_{m=1}^{N}\left[y^{(k)}_{[\vec{\mu}^{(k)}],m}\,z^{N-m}\right],$$

$$p_N^{(\vec{\mu}^{(k)};k)}\left(z;\vec{y}^{(k)}_{[\vec{\mu}^{(k)}]};\tilde{x}^{(k)}_{[\vec{\mu}^{(k)}]}\right)=\prod_{n=1}^{N}\left[z-x^{(k)}_{[\vec{\mu}^{(k)}],n}\right],$$

$$\vec{y}^{(k)}_{[\vec{\mu}^{(k)}]}=\vec{x}^{(k-1)}_{[\vec{\mu}^{(k)}]}=\tilde{x}^{(k-1)}_{[\vec{\mu}^{(k-1)}],[\mu_k]};\quad y^{(k)}_{[\vec{\mu}^{(k)}],m}=x^{(k-1)}_{[\vec{\mu}^{(k)}],m}\,.$$

The *k*-vector $\vec{\mu}^{(k)}=(\mu_1,\mu_2,\dots,\mu_k)$ has *integer* components μ_1, μ_2,…μ_k taking values in the range $1\leq\mu_1,\mu_2,\dots,\mu_k\leq N!$. The label μ_k identifies the permutation of the *a priori unordered* set $\tilde{x}^{(k-1)}_{[\vec{\mu}^{(k-1)}]}$ the components of which are the *N* zeros of the polynomial of the (*k-1*)-th generation with indices $\mu_1,\mu_2,\dots,\mu_{k-1}$, i. e. the *N* zeros of $p_N^{(\vec{\mu}^{(k-1)};k-1)}\left(z;\vec{y}^{(k-1)}_{[\vec{\mu}^{(k-1)}]};\tilde{x}^{(k-1)}_{[\vec{\mu}^{(k-1)}]}\right)$.

Question: Why introduce this notion of generations of monic polynomials?
Reply: **Why not?** And note the possibility mentioned above to generate many *solvable* dynamical systems---including many-body problems of Newtonian type (“accelerations equal forces”)---associated with such polynomials.
Question to the audience: is this notion of *generations of polynomials* new? Has it already been investigated?

The Ulam problem

Definition of (monic) “peculiar” polynomial: $\boldsymbol{p_N^{(p)}(z)}$

A (monic) polynomial of degree *N* is **peculiar** if (an appropriate permutation of its) *N* zeros coincide with its *N* coefficients, or equivalently if it vanishes when its argument coincides with one of its *N* coefficients:

$$p_N^{(p)}(z) = z^N + \sum_{m=1}^{N} y_m \; z^{N-m} = \prod_{n=1}^{N} (z - y_n)\,,$$

$$p_N^{(p)}(y_n) = 0\,, \qquad n = 1, \dots, N\,.$$

This notion (not the name “peculiar”) was introduced by S. Ulam, see *A collection of mathematical problems*, Interscience, New York, 1960, p. 30 f.; and he raised the issue of how many such polynomials of degree *N* exist.

It is easily seen that this question can be refined by introducing two separate subcategories of peculiar polynomials, which we call ***truly peculiar*** polynomials ($p_N^{(tp)}(z)$) respectively ***zero peculiar*** polynomials ($p_N^{(zp)}(z)$). The ***truly peculiar*** polynomials are those *peculiar* polynomials which feature no vanishing coefficient (and of course no vanishing zero); the ***zero peculiar*** polynomials are those *peculiar* polynomials which possess one or more **vanishing** coefficient (and zero). The motivation for introducing this distinction is due to the following obvious formula:

$$p_N^{(zp)}(z) = z^k \, p_{N-k}^{(tp)}(z) \, , \quad k = 1,2, \dots N \, .$$

Hence one can only focus on the *truly peculiar* polynomials.

The question of how many ***truly peculiar*** polynomials exist was answered in 1966, but only in the ***real*** number context: Paul R. Stein, “On Polynomial Equations with Coefficients Equal to Their Roots”, Amer. Math. Monthly **73**, 272-274 (1996). He showed that there are no such polynomials with degree $N>4$, and found *all* these polynomials with degree $N=2,3,$respectively 4: there are just 1, 2 respectively 1 such *tp-polynomials*:

$$N=2: \quad p_2^{(tp)}(z)=z^2+z-2=(z-1)(z+2)\,;$$

$$N=3: \quad p_3^{(tp)}(z)=z^3+z^2-z-1=(z-1)(z+1)^2\ ,$$

$$z^3+y_1z^2+y_2z+y_3=(z-y_1)(z-y_2)(z-y_3)$$

$$y_1=y,\ \ y_2=-\frac{1}{y}\,,\quad y_3=\frac{1}{(y+1)}$$

$$y=SingleRealRootOf\,[2y\verb|^|3+2y\verb|^|2=1]\,.$$

$$N = 4: \quad p_4^{(tp)}(z) = z^4 + y_1 z^3 + y_2 z^2 + y_3 z + y_4$$
$$= (z - y_1)(z - y_2)(z - y_3)(z - y_4) \; ,$$
$$y_1 = 1, \;\; y_2 = y \, , \quad y_3 = \frac{1}{y} \; , \;\; y_4 = y_1 y_2 y_3$$
$$y = SingleRealRootOf\,[y^3 + 2y^2 + y = -1] \, .$$

Our findings with F. Leyvraz [16, 17]

Proposition. (i) The number $\nu_p(N)$ of *p-polynomials* is *N*!. (ii) For all values of *N≠4* the number $\nu_{zp}(N)$ of *zp-polynomials* is (*N-1*)!, and the number $\nu_{tp}(N)$ of *tp-polynomials* is *N*!-(*N-1*)!=(*N-1*) (*N-1*)!. (iii) In the exceptional *N=4* case $\nu_{tp}(4)$=17=4!-3!-1 and $\nu_{zp}(4)$=7=3!+1 (with a *zp-polynomial* counted twice because of its multiplicity: see below). This of course implies that the rule

$$\nu_p(N) = \nu_{tp}(N) + \nu_{zp}(N)$$

holds for all *N*; while for *N*=2, 3, 4, 5, 6... ,

$$\nu_{tp}(N) = 1, 4, 17, 96, 600, ...$$

(recall that, as reported above, in the real context the corresponding numbers are 1, 2, 1, 0, 0 ,…) . □

Remark. Let us reemphasize that, in counting these polynomials, account must be taken of their multiplicity, i. e. of the fact that some of them may coincide: in the same sense as in the formulation of the fundamental theorem of algebra which states that a polynomial of degree *N* features precisely *N* zeros provided account is taken of the multiplicity of each of these zeros, in the exceptional cases of polynomials featuring multiple zeros. (But for a clarification in our context of the notion of the multiplicity of *p-polynomials*, and examples for *N=1,2,3,4,5*, see [17] and [18]; and note also that these findings contain a conjectural element…).

Dear young Jenya
best wishes from an
over-octogenarian.
Getting old is not too
bad --- better than
the alternative !!!