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Structure of coherent vortices generated by the inverse cascade of 2d turbulence in a finite box

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We consider the turbulence that is excited by an external pumping in a thin fluid layer. It can be treated as two-dimensional $(2d)$ on scales larger than the layer thickness. The pumping correlation length l is assumed to be much larger than the layer thickness, however, it should be much less that the box size L. The Reynolds number is assumed to be large.

The pumped $2d$ turbulence is described by the equation for vorticity ω

$$\partial_t \omega + \boldsymbol{v}\nabla\omega = \nabla \times \boldsymbol{f} + \nu\nabla^2\omega - \alpha\omega,$$

where $\boldsymbol{v}$ is velocity, $\boldsymbol{f}$ is pumping force per unit mass, ν is viscosity and α is bottom friction coefficient. The pumping force is assumed to be correlated at a scale $l \ll L$ and to be random in time.

There are two quadratic dissipationless integrals of motion, energy and enstrophy:

$$\int dx\ dy\ v^2, \qquad \int dx\ dy\ \omega^2.$$

Pumped turbulence – two cascades: enstrophy flows to small scales whereas energy flows to large scales, being dissipated by viscosity and friction, respectively (Kraichnan 1967, Leith 1968, Batchelor 1969).

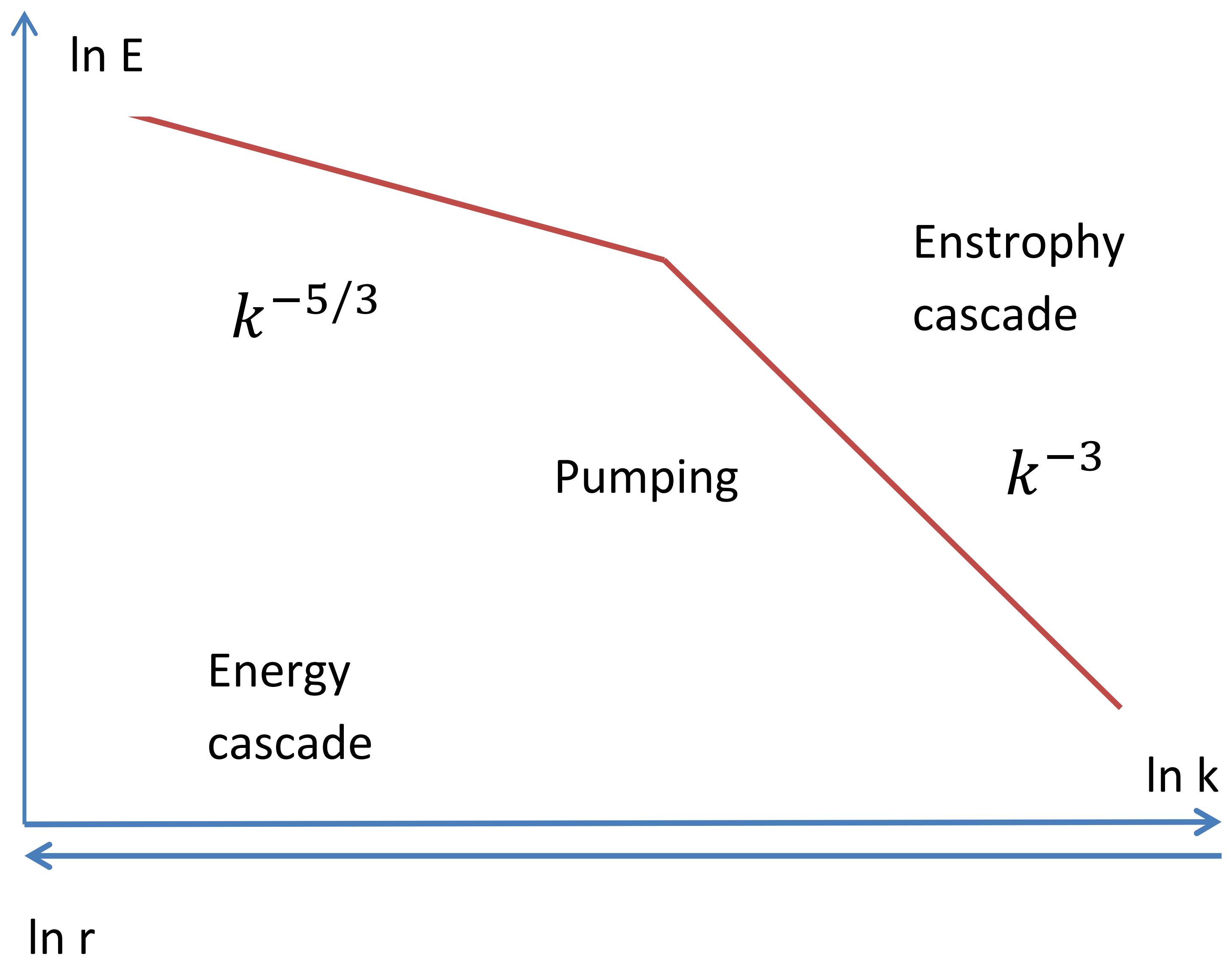
Constancy of the energy and enstrophy fluxes imply the proportionality laws

$$\langle (v_1 - v_2)\omega_1\omega_2 \rangle \propto r, \quad r \ll l;$$
$$\langle |v_1 - v_2|^3 \rangle \propto r, \quad r \gg l.$$

Suggest the normal scaling $v_1 - v_2 \propto r$ in the direct cascade and $v_1 - v_2 \propto r^{1/3}$ in the inverse cascade. The spectrum

$$\langle v_1 v_2 \rangle = \int \frac{dk}{2\pi} e^{ikr} E(k),$$

Then $E(k) \propto k^{-3}$ for the direct (enstrophy) cascade $E(k) \propto k^{-5/3}$ for the inverse (energy) cascade. Direct cascade – logarithmic correlation functions of vorticity (Falkovich, Lebedev 1994). Inverse cascade – an absence of anomalous scaling (Paret and Tabeling 1998, Boffetta, Celani and Vergassola 2000).



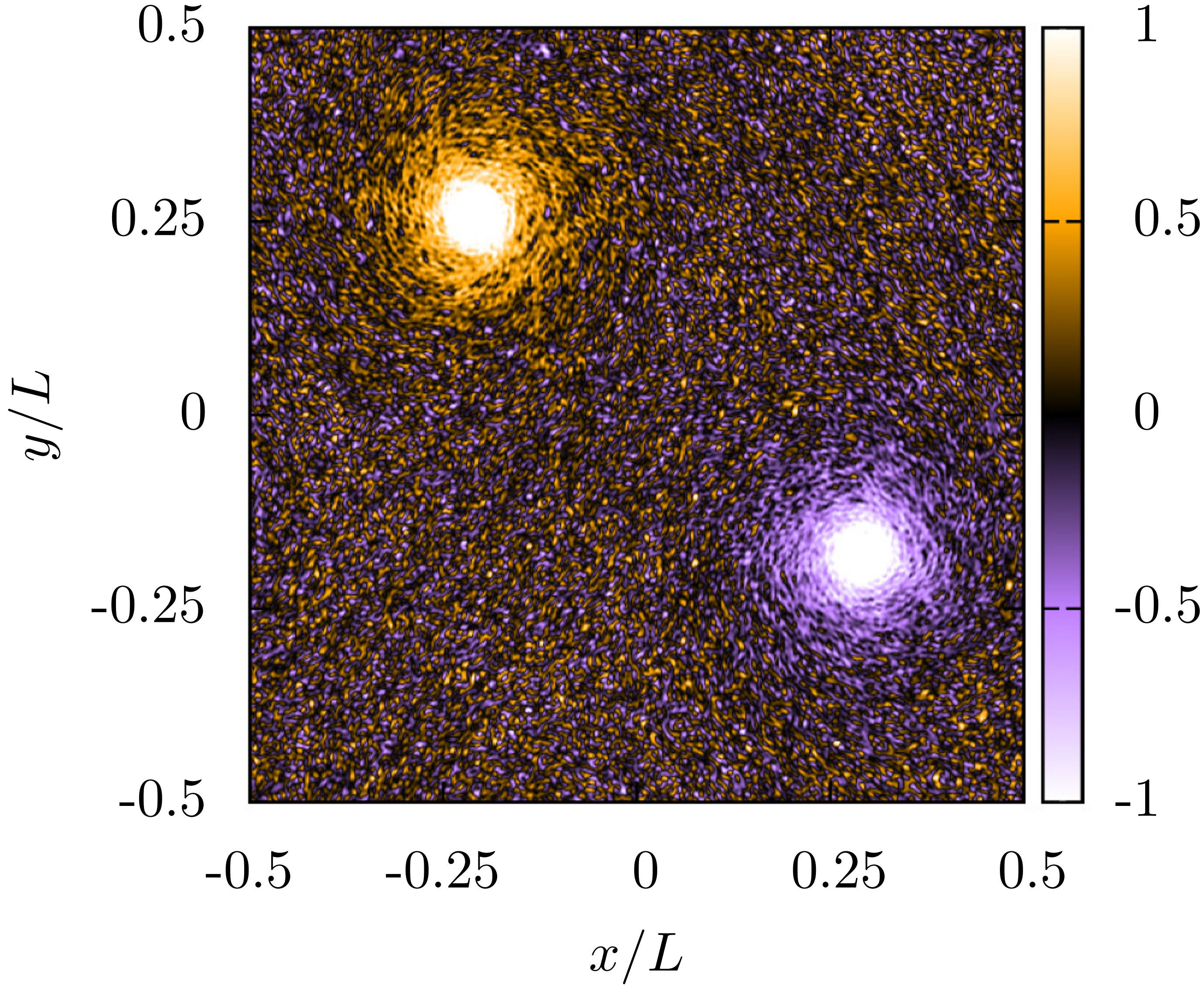
ln E
$k^{-5/3}$
Enstrophy
cascade
Pumping
k^{-3}
Energy
cascade
ln k
ln r

In an infinite system the inverse cascade is terminated by the friction at the scale $L_\alpha \sim \epsilon^{1/2}\alpha^{-3/2}$ where ϵ is the energy production rate per unit mass. If the box size $L < L_\alpha$ then the energy accumulates at L: experiment (Shats, Xia, Punzmann and Falkovich 2007) and numerics (Chertkov, Connaughton, Kolokolov and Lebedev 2007). Coherent structures are formed!

The coherent velocity profile arises at a time $t \sim t_L = L^{2/3}\epsilon^{-1/3}$. Movie After that the major part of the pumped energy is accumulated at scales $\sim L$. Therefore typical large-scale velocity $\sim \sqrt{\epsilon t}$ increases as time grows. The stage is terminated at time $t \sim \alpha^{-1}$. After that, at $t \gg \alpha^{-1}$, some steady (statistically homogeneous in time) state is realized. Movie

It is characterized by an average flow $\boldsymbol{V}$ and fluctuations on this background. As we saw, the average flow in the periodic setup consists of two coherent vortices. In laboratory experiments in a square box the average flow consists typically of five vortices: a big vortex in the center of the box and four counter-rotating smaller vortices in the corners of the box.

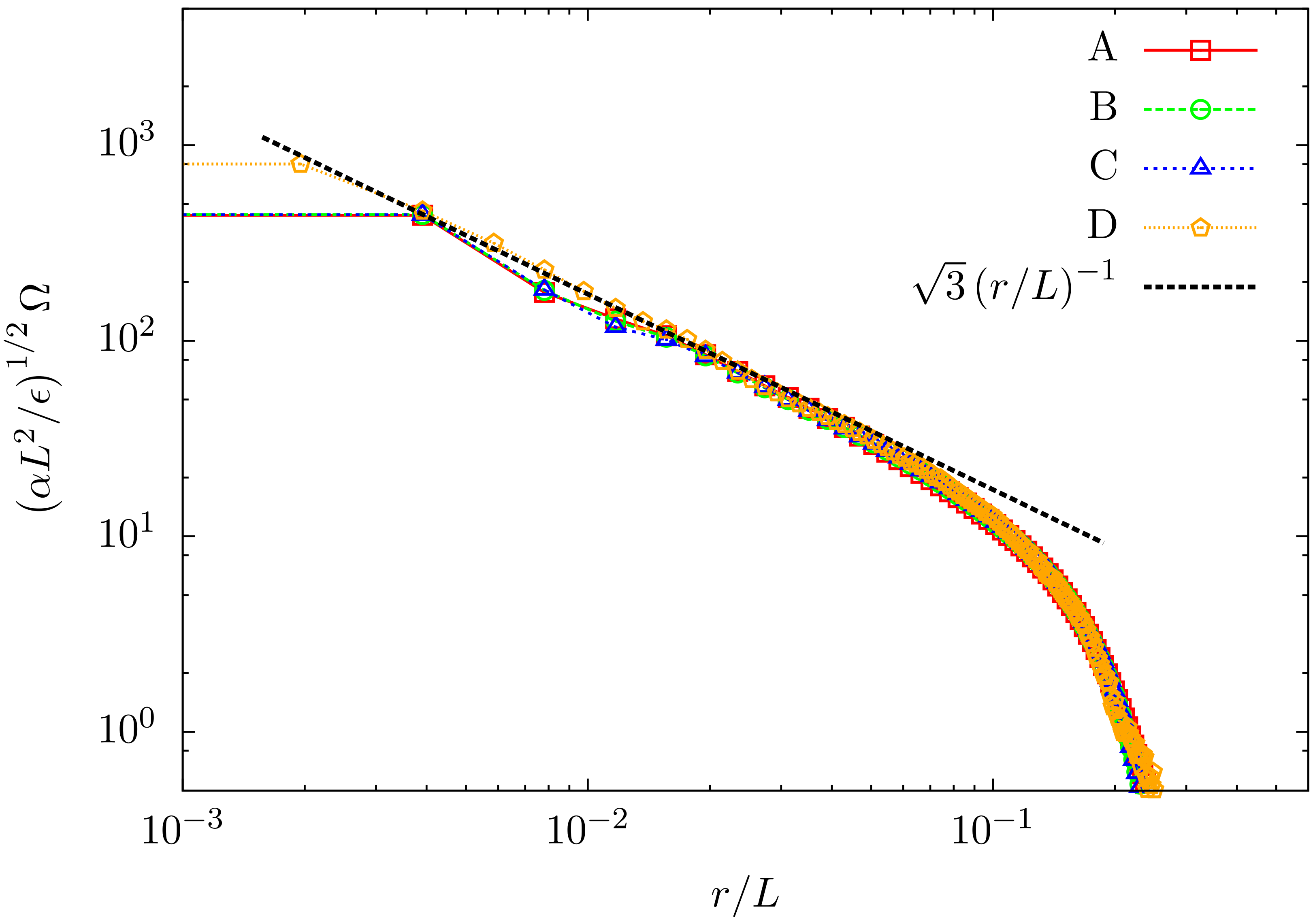
To extract properties of the average flow inside the vortex, one places origin at the vortex center. Both, experiment and numerics, show that the vortices are isotropic in average: the average polar velocity U and the average vorticity Ω are functions of the separation from the vortex center r. Snapshot of numerics Laurie, Bofetta, Falkovich, Kolokolov, Lebedev 2014.

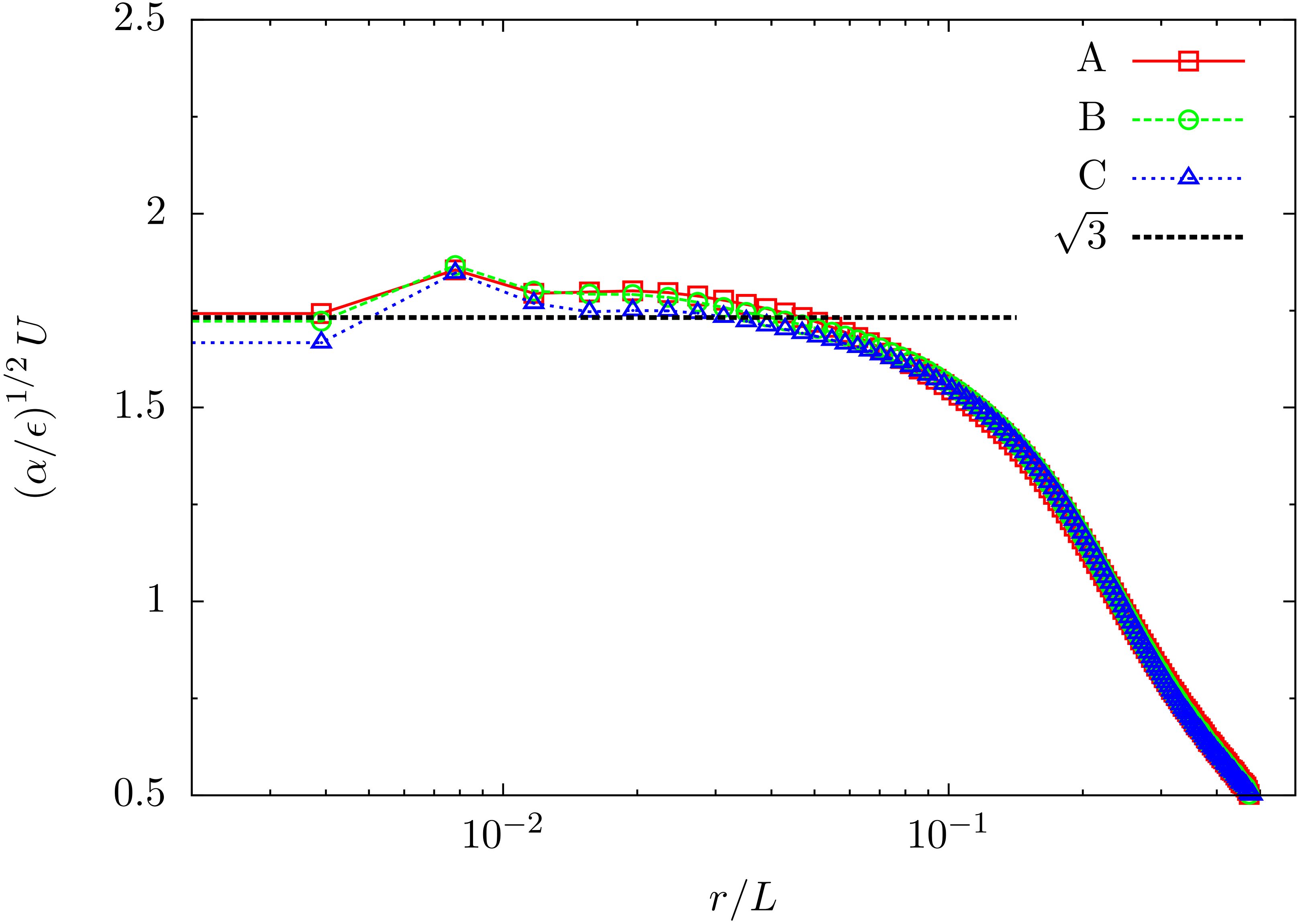


0.5
0.25
0
-0.25
-0.5
y/L
-0.5
-0.25
0
0.25
0.5
x/L
1
0.5
0
-0.5
-1

There is the hyperbolic region where the average velocity $\boldsymbol{V}$ is estimated as $\sqrt{\epsilon/\alpha}$ and the average vorticity Ω is estimated as $\sqrt{\epsilon/\alpha}/L$ from the energy balance: pumping versus friction. However, Ω increases toward the vortex center. One observes power-like behavior of Ω outside a relatively small core. The core radius is determined by viscosity.

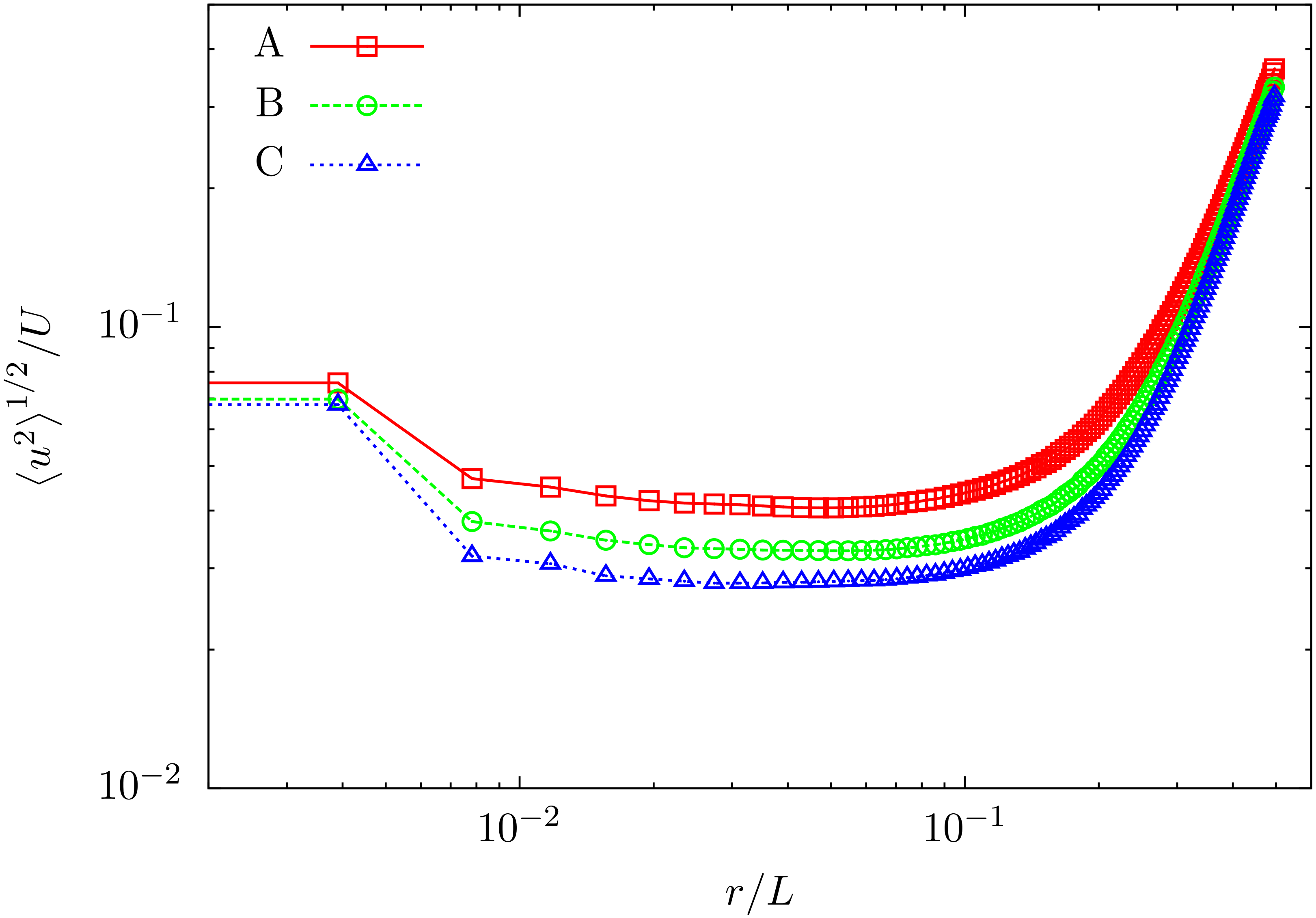
The numerics (Laurie, Bofetta, Falkovich, Kolokolov, Lebedev 2014), where pumping was short correlated in time, reveals the universal behavior U = const and $\Omega \propto r^{-1}$ outside the viscous core. The law is correct provided $r \ll L$ where L is the box size. Inside the core the average vorticity Ω is saturated and the average velocity U tends to zero $U \propto r$.

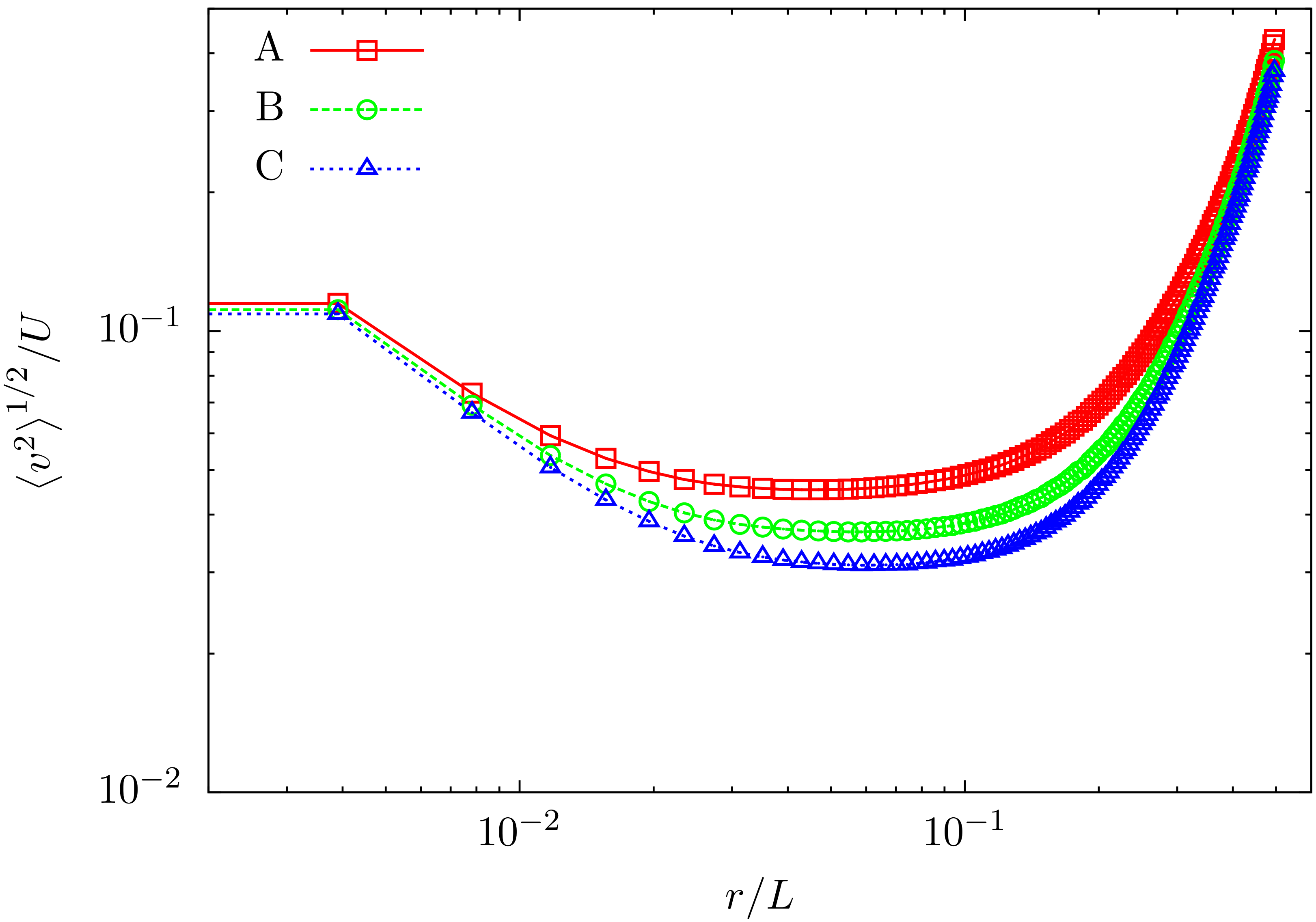

A
B
C
D
$\sqrt{3}\,(r/L)^{-1}$
10^{3}
10^{2}
10^{1}
10^{0}
$(\alpha L^2/\epsilon)^{1/2}\,\Omega$
10^{-3}
10^{-2}
10^{-1}
r/L



2.5
2
1.5
1
0.5
$(\alpha/\epsilon)^{1/2}U$
10^{-2}
10^{-1}
r/L
A
B
C
$\sqrt{3}$

In the universal region $u, v \ll U$, where u, v are polar and radial components of the fluctuating velocity. It is a consequence of the large value of the mean velocity gradient $\sim U/r$, growing toward the center of the vortex. The relative strength of fluctuations increases as r grows and on the periphery where $r \sim L$, fluctuations become stronger than inside the vortex.


A
B
C
10^{-1}
10^{-2}
$\langle u^2 \rangle^{1/2}/U$
10^{-2}
10^{-1}
r/L


A
B
C
10^{-1}
10^{-2}
$\langle v^2 \rangle^{1/2}/U$
10^{-2}
10^{-1}
r/L

Averaging the basic NS equation, one obtains the following equation for the vortex velocity

$$\alpha U = -\left(\partial_r + \frac{2}{r}\right)\langle uv\rangle + \nu\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)U$$

Comparing the terms in the equation one finds the core radius $R_c \sim (\nu/\alpha)^{1/2}$. To establish the flow profile outsize the core one has to find the average $\langle uv\rangle$ entering the equation for U.

In the work (Laurie, Bofetta, Falkovich, Kolokolov, Lebedev 2014) we derived the value of the average velocity in the universal region $U = \sqrt{3\epsilon/\alpha}$, where ϵ is the energy production rate per unit mass. The expression was derived from the conservation laws where some terms were neglected partly by symmetry partly by heuristic arguments. It is in a good agreement with numerics.

Further investigation (Kolokolov. Lebedev, 2016) shown that inside the universal region the flow fluctuations interact weakly and can be described in terms of the passive equation

$$\partial_t \varpi + (U/r)\partial_\varphi \varpi + v\partial_r \Omega = \phi - \hat{\Gamma}\varpi,$$

where ϖ is the fluctuating vorticity and $\hat{\Gamma}$ includes both bottom friction and viscosity. We solve the equation and then find $\langle uv \rangle$.

Direct calculations show that $\langle uv \rangle = 0$ if $\widehat{\Gamma} = 0$. The property is explained by symmetry reasoning: if $\widehat{\Gamma} = 0$ then the dynamic equations are invariant under the following combined transformation (kept U untouched)

$$t \to -t,\ \varpi \to \varpi,\ \varphi \to -\varphi,\ r \to r,\ v \to -v,\ u \to u.$$

Obviously, the average $\langle uv \rangle$ changes its sign at the transformation and is zero.

Next: if we take a nonzero $\widehat{\Gamma}$ then $\langle uv \rangle$ becomes nonzero and independent of $\widehat{\Gamma}$ (a kind of dissipation anomaly):

$$\langle uv \rangle = \epsilon / \Sigma,$$

where Σ is the local shear rate $\Sigma = \partial_r U - U/r$. Note that $\langle uv \rangle$ is gained at small scales where viscosity comes into game. There dissipation rate balances Σ.

Thus we arrive at the equation

$$\alpha U = -\left(\partial_r + 2/r\right)\epsilon/\Sigma,$$

that has a solution

$$U = \sqrt{3\epsilon/\alpha}, \qquad \Sigma = -U/r.$$

The solution is correct outside the vortex core. What is the restriction from the other side?

Our consideration was passive. The applicability condition of the approach is

$$r/U \ll l^{2/3}\epsilon^{-1/3},$$

where the right-hand side is characteristic turnover time at the pumping scale. Equating the quantities we find the radius of the universal zone $R_u \sim \epsilon^{1/6}\alpha^{-1/2}l^{2/3}$. What is outside the radius R_u?

At $r > R_u$ the situation is not passive, the direct cascade is restored and partly the inverse cascade does. Thus, nonlinearity is relevant and no consistent calculations are possible. Therefore we should apply the symmetry reasoning. The above symmetry is true for the non-linear dissipationless case as well. Therefore $\langle uv \rangle$ is gained at the viscous scale as previously.

The dissipation rate at the viscous scale can be estimated as $\gamma \sim \epsilon^{1/3}/l^{2/3}$, upto weak logarithmic corrections. One would expect the answer $\langle uv \rangle \sim \epsilon/\gamma$. However, it is absent due to isotropy of the direct cascade. Therefore we find

$$\langle uv \rangle \sim \epsilon^{1/3} \Sigma l^{4/3}.$$

where an additional factor Σ/γ is introduced accounting for slightly disturbed isotropy.

Substituting the result into the equation for U, one finds, that the average vorticity decreases exponentially outside the scale R_u. Thus R_u can be called the vortex radius. In Chertkov,Connaughton, Kolokolov, Lebedev 07 it is larger than the box size, in Laurie, Bofetta, Falkovich, Kolokolov, Lebedev 14 it is of the order of $1/10$ of the box size L. What will be if $R_u \ll L$?

Then one should think in terms of general equations. There is an analog of $\langle uv \rangle$ in the general case that should balance the bottom friction. Since the analog of $\langle uv \rangle$ is suppressed outside the vortices, the balance cannot be achieved for small R_u. We expect appearing a lot of vortices in this case.

One can find the velocity fluctuations in the universal region

$$\langle v^2\rangle, \langle u^2\rangle \sim k_f r\epsilon/\Sigma \qquad \Gamma k_f r \ll \Sigma,$$
$$\langle v^2\rangle, \langle u^2\rangle \sim \epsilon/\Gamma \qquad \Gamma k_f r \gg \Sigma.$$

Any case, $\langle v^2\rangle, \langle u^2\rangle \gg \langle uv\rangle$. One should separately calculate the zero angular harmonic $\langle u_0^2\rangle \sim \epsilon/(\Gamma k_f r)$. At $r \sim R_u$ the estimate coincides with Falkovich 2016.

Let x_r, x_φ are separations in the radial and polar directions. The structure functions are roughly linear in x:

$$S_{uu} \sim \frac{k_f \epsilon}{\Sigma} \left[|x_r| + x_\varphi \,\mathrm{arctan}(x_\varphi/|x_r|) \right],$$

$$S_{vv} \sim \frac{k_f \epsilon}{\Sigma} \left[x_\varphi \,\mathrm{arctan} \frac{x_\varphi}{|x_r|} + 2|x_r| \ln \frac{r}{\sqrt{x_r^2 + x_\varphi^2}} \right],$$

$$S_{vu} \sim -\frac{k_f \epsilon}{\Sigma} x_r \,\mathrm{arctan}(x_\varphi/|x_r|),$$

where $|x_r, x_\varphi| \ll r, \Sigma/(\Gamma k_f)$.

What will be in the case of finite correlation time of pumping? Then the effectiveness of the pumping is less than in the short correlated time. We analyze now two cases. The first case of finite correlation time τ, $\tau\Sigma > 1$. Effective passive regime at scales larger than U^3/ϵ The second case is static pumping - experiment. In progress.