Wave Turbulence in particle Physics and Cosmology

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General Motivation

Field theories in the particle physics and cosmology are nonlinear PDEs with non-trivial linear modes. Linear modes are the particles, the nonlinear interactions are the inter-particle reactions.
Wave turbulence provides a tool for understanding the particle reactions, Bose-Einstein condensation and describe new non-equilibrium states similar to the Kolmogorov cascade in turbulence.
Example of an interesting problem: the Higgs mechanism is essentially about finding linear normal modes in a symmetry-broken system. How will the nonlinearity affect it?
In this work you will see the Bose-Einstein condensation of pions, KZ spectra of their momenta, nonlocal interaction.

Gelmann's σ-model, also discussed by Goldstone, Nambu and many others since. It is the most used scalar field model for explaining the spontaneous symmetry breaking:

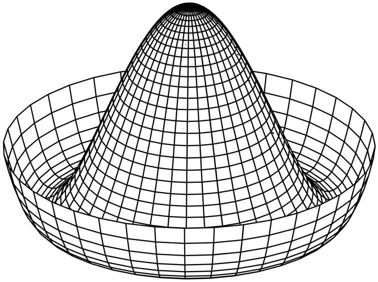
$$\psi_{tt} - \Delta\psi + (-1 + |\psi|^2)\psi = 0\,, \qquad (1)$$

where $\psi(\mathbf{x}, t) \in \mathbb{C}$. The linear part of this equation is like Klein-Gordon equation, but with a negative square mass ("superluminal particles"). No localised particle-like excitation around the base state $\psi = 0$ because it is unstable.
This is a model for interacting massive σ-meson and a massless pion (Goldstone boson).

Equation (1) conserves the total energy

$$E = \int \left[|\psi_t|^2 + |\nabla\psi|^2 - |\psi|^2 + \frac{1}{2}|\psi|^4 \right] d\mathbf{x} \,, \qquad (2)$$

where the last two terms correspond to the famous Mexican hat potential $U(\psi) = -|\psi|^2 + |\psi|^4/2$; we therefore refer to equation (1) as the Klein-Gordon Mexican Hat (KGMH) model.



Equation (1) is invariant w.r.t. $\psi \to e^{i\alpha}\psi$, with $\alpha =$ const. The invariant corresponding to this $U(1)$ symmetry is the "charge",

$$Q = \frac{i}{2}\int (\psi\partial_t\psi^* - \psi^*\partial_t\psi)\,\mathrm{d}\mathbf{x}\,. \tag{3}$$

In this work $Q = 0$.

Fix the symmetry broken state's phase to be 0, $\psi_0 = 1$, and consider perturbations of this state,

$$\psi(\mathbf{x}, t) = 1 + \phi(\mathbf{x}, t) \,.$$

Substitution into equation (1) yields

$$\phi_{tt} - \Delta\phi + \phi + \phi^* + 2\phi\phi^* + \phi^2 + |\phi|^2\phi = 0 \,. \tag{4}$$

We focus on the weakly-nonlinear dynamics, so neglect the cubic nonlinearity. In variables $\lambda = \Re(\phi)$ and $\chi = \Im(\phi)$ we have

$$\lambda_{tt} - \Delta\lambda + 2\lambda + 3\lambda^2 + \chi^2 = 0, \tag{5}$$

$$\chi_{tt} - \Delta\chi + 2\lambda\chi = 0. \tag{6}$$

Two linear modes $e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}$ with

$$\omega_{\mathbf{k}}^{a} = \sqrt{2+k^2}, \tag{7}$$
$$\omega_{\mathbf{k}}^{b} = k, \tag{8}$$

where $k = |\mathbf{k}|$. Branch a corresponds to massive particles whereas branch b – to a massless Goldstone boson. This is a basic model for coexisting σ-meson (branch a) and pion (branch b) fields, in which the pion's mass is neglected.

The system (5-6) follows from the following Hamiltonian,

$$H = \int \left[\frac{1}{2}p^2 + \frac{1}{2}q^2 + \frac{1}{2}(\nabla\lambda)^2 + \lambda^2 + \frac{1}{2}(\nabla\chi)^2 + \lambda^3 + \lambda\chi^2 \right] d\mathbf{x}\,,$$

where p and q are the momenta conjugate to the variables λ and χ respectively. Hamilton equations are

$$\lambda_t = \frac{\delta H}{\delta p}, \qquad p_t = -\frac{\delta H}{\delta \lambda}, \tag{9}$$

$$\chi_t = \frac{\delta H}{\delta q}, \qquad q_t = -\frac{\delta H}{\delta \chi}. \tag{10}$$

We consider these equations in a d-dimensional periodic cube of side L and introduce Fourier series,

$$\begin{pmatrix} \lambda(\mathbf{x},t) \\ \chi(\mathbf{x},t) \\ p(\mathbf{x},t) \\ q(\mathbf{x},t) \end{pmatrix} = \sum_{\mathbf{k}} \begin{pmatrix} \hat{\lambda}_{\mathbf{k}}(t) \\ \hat{\chi}_{\mathbf{k}}(t) \\ \hat{p}_{\mathbf{k}}(t) \\ \hat{q}_{\mathbf{k}}(t) \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad (11)$$

where the sums are over $\mathbf{k} \in \frac{2\pi}{L}\mathbb{Z}^d$. Because these four fields are real-valued, $(\hat{\lambda}_{-\mathbf{k}}, \hat{\chi}_{-\mathbf{k}}, \hat{p}_{-\mathbf{k}}, \hat{q}_{-\mathbf{k}}) = (\hat{\lambda}^*_{\mathbf{k}}, \hat{\chi}^*_{\mathbf{k}}, \hat{p}^*_{\mathbf{k}}, \hat{q}^*_{\mathbf{k}})$.

In Fourier space, Hamilton equations therefore become

$$\dot{\hat{\lambda}}_{\mathbf{k}} = \frac{\partial H}{\partial \hat{p}^*_{\mathbf{k}}}, \qquad \dot{\hat{p}}_{\mathbf{k}} = -\frac{\partial H}{\partial \hat{\lambda}^*_{\mathbf{k}}}, \tag{12}$$

$$\dot{\hat{\chi}}_{\mathbf{k}} = \frac{\partial H}{\partial \hat{q}^*_{\mathbf{k}}}, \qquad \dot{\hat{q}}_{\mathbf{k}} = -\frac{\partial H}{\partial \hat{\chi}^*_{\mathbf{k}}}, \tag{13}$$

with

$$H = H_2 + H_{int}, \tag{14}$$

$$H_2 = \frac{1}{2}\sum_{\mathbf{k}} \left[|\hat{p}_{\mathbf{k}}|^2 + (\omega^a_{\mathbf{k}})^2 |\hat{\lambda}_{\mathbf{k}}|^2 + |\hat{q}_{\mathbf{k}}|^2 + (\omega^b_{\mathbf{k}})^2 |\hat{\chi}_{\mathbf{k}}|^2 \right], \tag{15}$$

$$H_{int} = \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \left[\hat{\lambda}_{\mathbf{k}_1}\hat{\lambda}_{\mathbf{k}_2}\hat{\lambda}_{\mathbf{k}_3} + \hat{\lambda}_{\mathbf{k}_1}\hat{\chi}_{\mathbf{k}_2}\hat{\chi}_{\mathbf{k}_3} \right] \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3), \tag{16}$$

where δ is the Kroenecker symbol.

Introduce the normal variables $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ that diagonalize H_2,

$$\hat{\lambda}_{\mathbf{k}} = \frac{a_{\mathbf{k}} + a^*_{-\mathbf{k}}}{\sqrt{2\omega^a_{\mathbf{k}}}}, \qquad \hat{p}_{\mathbf{k}} = \frac{\sqrt{\omega^a_{\mathbf{k}}}(a_{\mathbf{k}} - a^*_{-\mathbf{k}})}{i\sqrt{2}}, \tag{17}$$

$$\hat{\chi}_{\mathbf{k}} = \frac{b_{\mathbf{k}} + b^*_{-\mathbf{k}}}{\sqrt{2\omega^b_{\mathbf{k}}}}, \qquad \hat{q}_{\mathbf{k}} = \frac{\sqrt{\omega^b_{\mathbf{k}}}(b_{\mathbf{k}} - b^*_{-\mathbf{k}})}{i\sqrt{2}}. \tag{18}$$

We refer to $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ as the respective amplitudes of a and b waves.

In these variables the equations of motion are

$$i\dot{a}_{\mathbf{k}} = \frac{\partial H}{\partial a^*_{\mathbf{k}}}, \quad (19)$$

$$i\dot{b}_{\mathbf{k}} = \frac{\partial H}{\partial b^*_{\mathbf{k}}}, \quad (20)$$

where $H = H_2 + H_{int}$, with

$$H_2 = \sum_{\mathbf{k}} \left[\omega^a_{\mathbf{k}} |a_{\mathbf{k}}|^2 + \omega^b_{\mathbf{k}} |b_{\mathbf{k}}|^2 \right], \quad (21)$$

$$\begin{aligned} H_{int} = \quad & \frac{1}{2\sqrt{2}} \sum\nolimits_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \frac{1}{\sqrt{\omega^a_1 \omega^a_2 \omega^a_3}} \left[(a_1 a_2 a_3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \right. \\ & \left. + 3 a_1 a_2 a^*_3 \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3)) \right] \\ & + \frac{1}{2\sqrt{2}} \sum\nolimits_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \frac{1}{\sqrt{\omega^a_1 \omega^b_2 \omega^b_3}} \left[a_1 b_2 b_3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \right. \\ & \left. + 2 a_1 b_2 b^*_3 \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3) + a^*_1 b_2 b_3 \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \right] + c.c.. \end{aligned} \quad (22)$$

The terms in the Hamiltonian (22) correspond to 3-wave processes of type

$$a + a + a \to 0,\ a + a \to a,\ a + b + b \to 0,\ a + b \to b,\ b + b \to a.$$

Out of these processes, the only one allowed by the frequency resonance condition is $b + b \to a$, so can discard all the other terms in H_{int} and write

$$H_{int} = \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} V^1_{23} a^*_1 b_2 b_3 \delta^3_{12} + c.c., \qquad (23)$$

where we have a shorthand notation of Kronecker deltas, $\delta^1_{23} \equiv \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)$, and the interaction coefficient

$$V^1_{23} \equiv \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{\omega^a_1 \omega^b_2 \omega^b_3}}. \qquad (24)$$

The equations of motion become

$$\dot{a}_{\mathbf{k}} = -i\omega^a_{\mathbf{k}} a_{\mathbf{k}} - i \sum_{\mathbf{k}_1,\mathbf{k}_2} V^{\mathbf{k}}_{12}\, b_1 b_2 \delta^{\mathbf{k}}_{12}, \tag{25}$$

$$\dot{b}_{\mathbf{k}} = -i\omega^b_{\mathbf{k}} b_{\mathbf{k}} - 2i \sum_{\mathbf{k}_1,\mathbf{k}_2} V^{1}_{2\mathbf{k}}\, a_1 b^*_2 \delta^{1}_{2\mathbf{k}}. \tag{26}$$

Note: a Hamiltonian system with two types of interacting waves was studied by Zakharov, Musher and Rubenchik in 1985. In their study the dominant process was of the type $a + b \to b$, which leads to very different dynamics than the process $b + b \to a$ considered here.

Define the wave action spectra of the modes a and b:

$$\lim_{L\to\infty} n^a_{\mathbf{k}} = \left(\frac{L}{2\pi}\right)^d \langle |a_{\mathbf{k}}|^2 \rangle\,, \quad \lim_{L\to\infty} n^b_{\mathbf{k}} = \left(\frac{L}{2\pi}\right)^d \langle |b_{\mathbf{k}}|^2 \rangle\,,$$

Assuming randomness of the initial phases and amplitudes, taking $L \to \infty$ and $a_{\mathbf{k}}, b_{\mathbf{k}} \to 0$ limits, using the standard wave turbulence approach we obtain the following kinetic equations,

$$\dot{n}^a_{\mathbf{k}} = 4\pi \int |V^{\mathbf{k}}_{12}|^2 \,(n^b_1 n^b_2 - 2 n^b_1 n^a_{\mathbf{k}}) \delta(\omega^{abb}_{\mathbf{k}12}) \delta^{\mathbf{k}}_{12}\, d\mathbf{k}_1 d\mathbf{k}_2, \quad (27)$$

$$\dot{n}^b_{\mathbf{k}} = 8\pi \int |V^1_{2\mathbf{k}}|^2 \,(n^a_1 n^b_2 + n^a_1 n^b_{\mathbf{k}} - n^b_2 n^b_{\mathbf{k}}) \delta(\omega^{abb}_{12\mathbf{k}}) \delta^1_{2\mathbf{k}}\, d\mathbf{k}_1 d\mathbf{k}_2. \quad (28)$$

Kinetic equations (27) and (28) conserve the total wave energy,

$$\mathcal{E} = \int (\omega_{\mathbf{k}}^{a} n_{\mathbf{k}}^{a} + \omega_{\mathbf{k}}^{b} n_{\mathbf{k}}^{b})\, d\mathbf{k}, \tag{29}$$

as well as the following particle invariant,

$$\mathcal{N} = \int (2 n_{\mathbf{k}}^{a} + n_{\mathbf{k}}^{b})\, d\mathbf{k}. \tag{30}$$

This second invariant is interesting. The process $a \rightleftharpoons b + b$ is similar to a chemical reaction where a is a "molecule" made of two b "atoms", and $\mathcal{N}$ would be the total number of such b "atoms": 1 per b wave, and 2 per a wave.

One can easily check that the kinetic equations admit solutions corresponding to the equilibrium Rayleigh-Jeans (or thermodynamic) equilibrium given by

$$n_{\mathbf{k}}^{a} = \frac{T}{\omega_{\mathbf{k}}^{a} + 2\mu}, \quad n_{\mathbf{k}}^{b} = \frac{T}{\omega_{\mathbf{k}}^{b} + \mu}, \tag{31}$$

for any temperature $T =$ const and chemical potential $\mu =$ const. The RJ states is expected in a truncated Fourier space. However, it does not form for some initial conditions: Bose-Einstein condensation occurs for low $F = E/N$.

No local KZ solutions exist in this case!

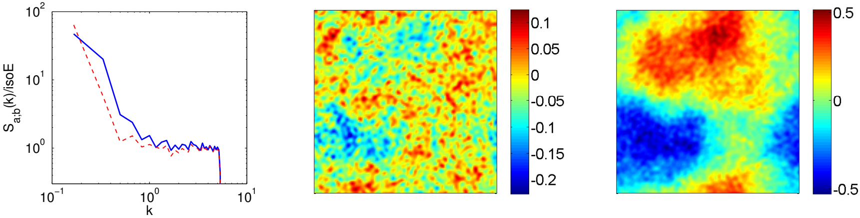


Figure: $\mathcal{F}$ is below the BEC threshold. Left: spectrum of a and b waves normalized by energy equipartition spectra. The flat part on the rhs is in thermal equilibrium, and the bump at low k indicates condensation. Centre and right: snapshots at final time of the λ and χ fields (modes a and b respectively).

Condensate fraction

$$n_k^a = \mathcal{N}_a^{(cond)}\,\delta(k) + \tilde{n}_k^a\,, \quad (32)$$

$$n_k^b = \mathcal{N}_b^{(cond)}\,\delta(k) + \tilde{n}_k^b\,. \quad (33)$$

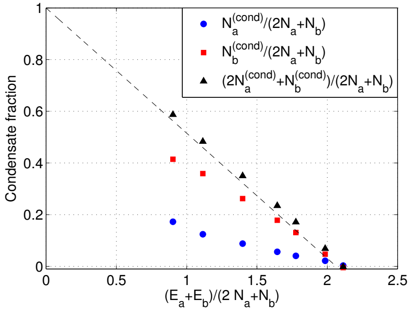


Figure: Condensate fraction as a function of the energy per particle, $\mathcal{F} = \mathcal{E}/\mathcal{N}$. We show the condensate fraction of a particles (•), b particles (□), and $2\mathcal{N}_a + \mathcal{N}_b$ (△). The dashed-line is the theoretical prediction.

Dynamical condensation as a nonlocal KZ cascade

Now let us consider a dynamical process by which condensation is achieved in an *untruncated* system. First, one can show that ensembles of b-waves at low-enough wave numbers are steady solutions to the kinetic equations. We refer to such solutions as "pion condensates". We consider a small perturbation of the a- and b-wave spectra and study the evolution of these perturbations through a new effective kinetic equation.

Dynamical condensation as a nonlocal KZ cascade

At low $\mathcal{F} = \mathcal{E}/\mathcal{N} \ll \mathcal{F}_{crit}$, most of the particles are in the b-condensate, and the remaining uncondensed particles are in both a and b modes,

$$n_k^a = \tilde{n}_k^a\,, \tag{34}$$
$$n_k^b = A\delta(k) + \tilde{n}_k^b\,. \tag{35}$$

Inserting this into the kinetic equations for leads to

$$\dot{\tilde{n}}_k^a = \frac{4\pi A}{k\sqrt{2+k^2}}\left(\tilde{n}^b_{\sqrt{2+k^2}} - \tilde{n}_k^a\right) + \frac{2\pi}{k\omega_k^a}\int \delta(\omega^{abb}_{k12})\,(\tilde{n}_1^b\tilde{n}_2^b - 2\tilde{n}_1^b\tilde{n}_k^a)\,dk_1\,dk_2,$$
$$\dot{\tilde{n}}_k^b = \frac{4\pi A}{k^2}\left(\tilde{n}^a_{\sqrt{k^2-2}} - \tilde{n}_k^b\right) + \frac{4\pi}{k^2}\int \frac{k_1}{\omega_1^a}\delta(\omega^{abb}_{12k})\,(\tilde{n}_1^a\tilde{n}_2^b + \tilde{n}_1^a\tilde{n}_k^b - \tilde{n}_2^b\tilde{n}_k^b)\,dk_1\,dk_2.$$

Note that there is an exact energy-equipartition ($\mu = 0$) solution for $\tilde{n}_k^a$ and $\tilde{n}_k^b$ in perfect agreement with the thermodynamic treatment of Bose-Einstein condensation.

Dynamical condensation as a nonlocal KZ cascade

In the leading order in $\tilde{n}/A$ we can drop the integrals. The remaining equations describe approaching, after a short transient, to an equilibrium:

$$\tilde{n}^{a;0}_{\sqrt{k^2-2}} = \tilde{n}^{b;0}_k \equiv n_k \,. \tag{36}$$

In the next order we have for n_k and equation which, for $k \gg 1$, is: In the relativistic limit $k \gg 1$ the equation becomes scale-invariant,

$$\begin{aligned} \dot{n}_k &= \frac{\pi}{k^2}\left[\int_{k_1 \le k} n_{k_1} n_{k-k_1} - 2 n_{k_1} n_k \, dk_1 \right. \\ &\left. +2 \int_{k_1 \ge k} n_{k_1} n_{k_1-k} + n_{k_1} n_k - n_{k_1-k} n_k \, dk_1 \right] . \end{aligned} \tag{37}$$

This equation has an exact solution for the KZ type (direct energy cascade):

$$n_k = \frac{\mathcal{P}^{\frac{1}{2}} \, k^{-\frac{3}{2}}}{4\pi\sqrt{2\pi - 8\ln 2}} \,. \tag{38}$$

Numerical simulation of nonequilibrium condensation

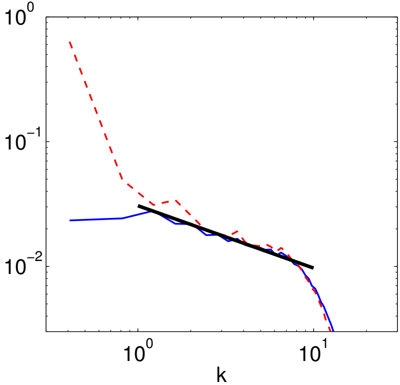


Figure: Out-of-equilibrium spectra in a numerical simulation of the KGMH equation. Most of the massive waves (σ-mesons) decay into massless waves (pions). A strong pion condensate of appear. The remaining small-scale spectra are in quantitative agreement with the KZ prediction (38), shown as a thick solid line - including the prefactor!

Summary

- Nonlinear σ-model describes interactions of the massive waves (σ-mesons) decay into massless waves (pions).
- Symmetry breaking and cosmological phase transitions.
- Wave turbulence predicts condensation of Goldstone bosons (pions).
- Over-condensate excitations are described by a 3-wave kinetic equation.
- This kinetic equation admits an energy cascade KZ solution. In a infinite system, this solution corresponds to a condensate that increases in amplitude, while the energy cascades to larger and larger wave numbers. In a truncated system, the cascade is eventually arrested and the system reaches thermodynamic equilibrium.
- Next step - add vector field and study the nonlinear Higgs effect.