ELASTIC SHEETS, PHASE SURFACES and PATTERN UNIVERSES

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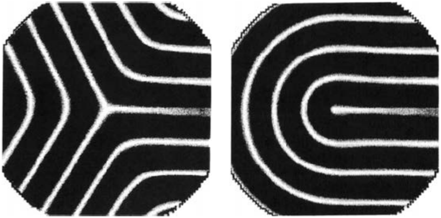
- Connecting the geometry of energy functionals of elastic sheets and patterns
- Condensation of Gaussian curvature on point and loop defects. Topological indices of “spin” and “charge”.
- Parallels between pattern universes and fundamental particle physics and cosmology. Quarks, leptons, inflation, dark matter, dark energy.

http://arxiv.org/abs/1612.01007

$$
\begin{aligned}
E &= E_s + E_b \\
&= h \int_\Omega \mu \left((E-1)^2 + 2F^2 + (G-1)^2 \right) + \frac{\lambda}{2} (E + G - 2)^2 \cdot d\vec{x} \\
&\quad + \frac{h^3}{3} \int_\Omega \mu \left(L^2 + 2M^2 + N^2 \right) + \frac{\lambda\mu}{\lambda + 2\mu} (L + N)^2 \cdot d\vec{x} \\
&\quad + O \left(||E-1, F, G-1|| h^3 \right)
\end{aligned}
$$

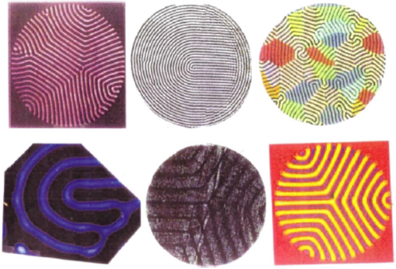
Metric 2-form	$E dx_1^2 + 2F dx_1 dx_2 + G dx_2^2$
Curvature 2-form	$L dx_1^2 + 2M dx_1 dx_2 + N dx_2^2$

Patterns and their point defects



(a) Concave disclination. (b) Convex disclination.

Point defects in patterns



Concave-convex point disclinations in experiments.

Microscopic

$$\omega_t = R\omega - \left(\nabla^2 + 1\right)^2 \omega - \omega^3 = -\frac{\delta E}{\delta \omega}$$

$$E = \int \left(\frac{1}{2} \left(\left(\nabla^2 + 1\right) \omega \right)^2 - \frac{1}{2} R\omega^2 + \frac{1}{4}\omega^4 \right) dxdy$$

$$\omega = f\left(\theta = \vec{k} \cdot \vec{x}; \{A_n\}, R\right) = \sum A_n(k, R) \cos n\theta$$

Exact 2π solution

$$\omega = f\left(\theta; \{A_n\}, R\right) + \text{corrections}$$

$$\nabla\theta = \vec{k} \text{ slowly varying : } \varepsilon = \frac{\lambda}{L}$$

$$< f_\theta^2 > \frac{\partial \theta}{\partial t} = -\nabla \cdot \vec{k} B(k) - < f_\theta^2 > \nabla^4 \theta = -\frac{\delta \overline{E}}{\delta \theta}$$

$$B(k) = -\frac{1}{2} \frac{d}{dk^2} < \omega^4 >$$

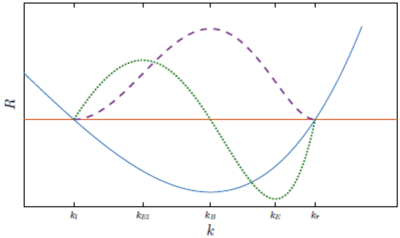


Figure 2: The graphs of $\langle w^4 \rangle$ (dashed) and $kB(k)$ (dotted) superposed on the neutral stability curve (solid) of the uniform $w = 0$ solution of (8). The horizontal line corresponding to the given stress parameter R intersects the neutral stability curve at wavenumbers k_l and k_r. The uniform state is linearly unstable to perturbations with wavenumbers $k_l < k < k_r$. The wavenumbers corresponding to the Eckhaus instability k_{El} and k_E, and the largest growth rate, k_0, are also indicated on the figure.

$$\overline{E} = \int \frac{1}{4} < \omega^4 > \Big|_{k^2}^{1} dxdy + \varepsilon^2 \int \Big\{ (\nabla \cdot \mathbf{k})^2 < \omega_\theta^2 >$$

$$+ 2(\nabla \cdot \mathbf{k})(\mathbf{k} \cdot \nabla k^2) \frac{d < \omega_\theta^2 >}{dk^2}$$

$$+ (\mathbf{k} \cdot \nabla)(\mathbf{k} \cdot \nabla) < \omega_\theta^2 > \Big\} dxdy$$

After times long w.r.t. horizontal diffusion time

$$\overline{E} = \int \frac{1}{4} < \omega^4 > \Big|_{k^2}^{1} dxdy$$

$$= \varepsilon^2 < \omega_\theta^2 > \int \Big((\nabla \cdot \vec{k})^2 - 2(\theta_{xx}\theta_{yy} - \theta_{xy}^2) \Big) dxdy$$

Phase surface $x, y, z = \frac{1}{k_0}\theta(x, y)$

Metric 2-form $E = 1 + \frac{\theta_x^2}{k_0^2}, \quad F = \frac{\theta_x \theta_y}{k_0^2}, \quad G = 1 + \frac{\theta_y^2}{k_0^2}$

Curvature 2-form $L = \frac{\theta_{xx}}{\sqrt{k^2 + k_0^2}}, \quad M = \frac{\theta_{xy}}{\sqrt{k^2 + k_0^2}}, \quad N = \frac{\theta_{yy}}{\sqrt{k^2 + k_0^2}}$

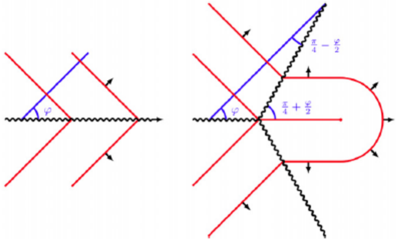
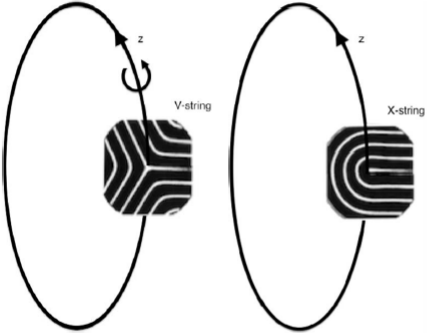


Figure 5: The nipple instability.



(a) A V-string. (b) A X-string.

Figure 8: Loop defects. z is the coordinate along the "backbone" and the pattern is periodic in z.

Pattern Dark Matter

$$\overline{E} = \frac{\rho_0 c^2}{k_0^4} \int \left\{ \left(|\nabla\theta|^2 - c^{-2}\theta_t^2 - k_0^2 \right)^2 + \varepsilon^2 \left(\Delta\theta - c^{-2}\theta_{tt} \right)^2 \right\} d^3x dt$$

Self dual $\quad + \quad \theta = \mp\varepsilon \ln \psi$

$$\frac{1}{c^2}\psi_{tt} - \Delta\psi + \frac{k_0^2}{\varepsilon^2} = 0$$

$$\psi(r) = \frac{Ae^{k_0 r/\varepsilon} + Be^{k_0 r/\varepsilon}}{r}$$

$$\theta \simeq \begin{array}{cc} -k_0 r + \varepsilon \ln r & r \gg 1 \\ \theta_0 + \theta_2 r^2 & r \ll 1 \end{array}$$

Energy density

$$\rho c^2 = \rho_0 c^2 k_0^{-4} \left(2\varepsilon k_0 r^{-1} - \varepsilon^2 r^{-2} \right)^2 \qquad r \geq \frac{\varepsilon}{k_0}$$

$$\rho_0 c^2 O(1) \qquad r \leq \frac{\varepsilon}{k_0}$$

$$M = 4\pi \int \rho(r) r^2 dr$$

Balance $\dfrac{GM}{r^2}$ and $\dfrac{v^2}{r}$

$$\Rightarrow v \simeq \begin{matrix} \sqrt{4\pi G \rho_0} r & r \leq \frac{\varepsilon}{k_0} \\ \sqrt{4\pi G \rho_0} \frac{\varepsilon}{k_0} & r \geq \frac{\varepsilon}{k_0} \end{matrix}$$

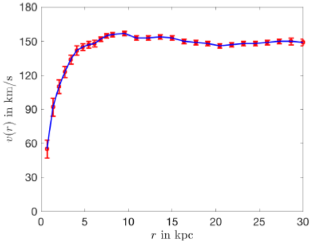


Figure 9: The rotation curve for NGC 3198. r is the distance from the galactic center and $v(r)$ is the rotation speed. The data is from van Albada *et al* [71].