High-order Korteweg-de Vries equations: physical applications and modulational instability

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May 21 - May 25, 2017

Korteweg-de Vries Equation (1895)

Diederik Johannes Korteweg
(1848-1941)

Gustav de Vries
(1866-1934)

Discovered
for water waves,
KdV equation
describes weakly
nonlinear and
weakly dispersive
waves in many
physical systems

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

Canonical form for unidirectional propagation
in reference system x – c t

Modified Korteweg-de Vries Equation

$$\frac{\partial u}{\partial t} \pm 6u^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

- is well-known in nonlinear mathematics (fully integrable model),

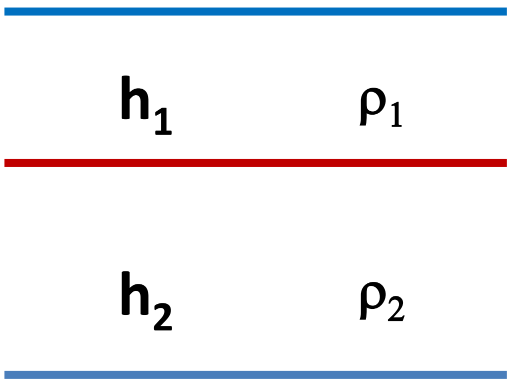
- less in nonlinear physics

T.L. Perelman, A.Kh. Fridman, M.M. Yelyashevich, Modified KdV equation in electro-dynamics. *Sov. Phys. JETP , 1974, vol.* 39, 643-646.

Pelinovskii E.N., and Sokolov V.V. Nonlinear theory for the propagation of electromagnetic waves in size-quantized films*. Radiophysics and Quantum Electronics*, 1976, vol. 19, N. 4, 378 -382.

$$v\frac{\partial \mathscr{E}}{\partial x} + \frac{\partial \mathscr{E}^3}{\partial \tau} + \frac{1}{\omega_0^2}\frac{\partial^3 \mathscr{E}}{\partial \tau^3} + \frac{2\nu}{\pi\omega_0}\int_{-\infty}^{\infty}\frac{\partial \mathscr{E}}{\partial \tau'}\frac{d\tau'}{\tau-\tau'} = 0.$$

KdV Equation for waves in 2-layer flow



$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0$$

$$c = \sqrt{\frac{g\Delta\rho}{\rho} \frac{h_1 h_2}{h_1 + h_2}}$$

$$\beta = \frac{c h_1 h_2}{6}$$

$$\alpha = \frac{3c}{2} \frac{h_1 - h_2}{h_1 h_2}$$

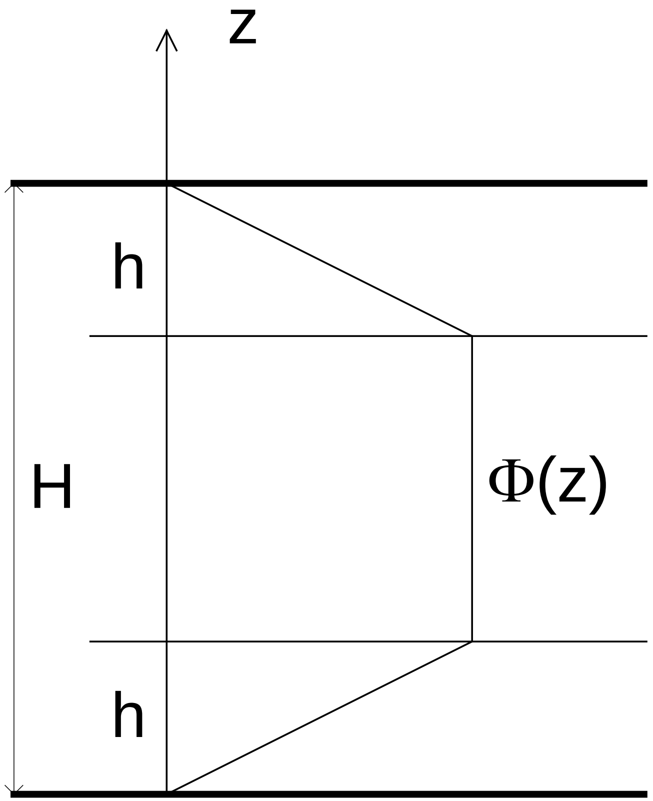
mKdV Equation for waves in 2-layer flow

If $h_1 = h_2$

$$\frac{\partial u}{\partial t} + \alpha_1 u^2 \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0$$

$$\alpha_1 = -\frac{3c}{h^2} < 0$$

mKdV Equation for waves in 3-layer flow

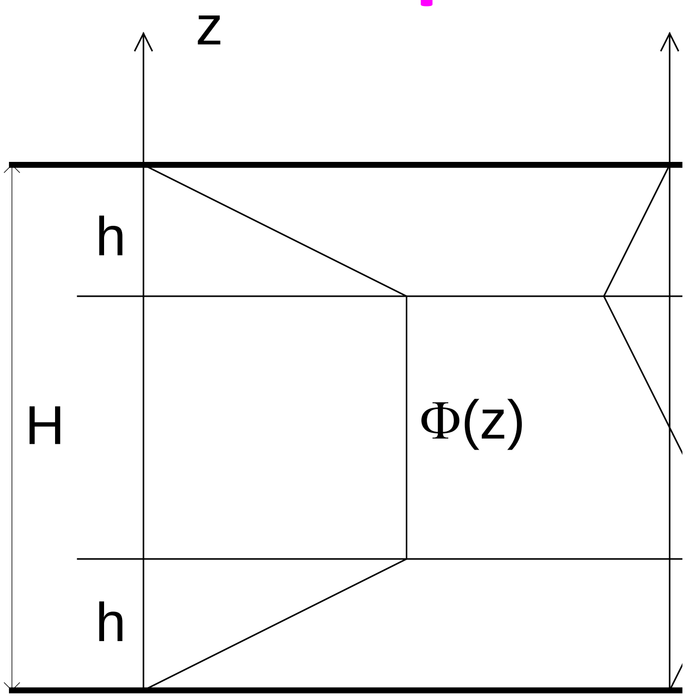


$$\frac{\partial u}{\partial t} + \alpha_1 u^2 \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0$$

$$\alpha_1 = -\frac{3c}{4h^2}\left(13 - \frac{9H}{2h}\right)$$

Grimshaw, R., Pelinovsky, E., and Talipova, T. The modified Korteweg - de Vries equation in the theory of large-amplitude internal waves. *Nonlinear Processes in Geophysics,* 1997, vol. 4, N. 4, 237 - 350

5^{th} KdV Equation for waves in 3-layer flow



$$\frac{\partial u}{\partial t} + \alpha_1 u^2 \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0$$

$$\alpha_1 = -\frac{3c}{4h^2}\left(13 - \frac{9H}{2h}\right)$$

If h = 9H/26

$$\frac{\partial u}{\partial t} + \alpha_3 u^4 \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0$$

$$\alpha_3 < 0$$

Kurkina O.E., Kurkin A.A., Soomere, T. Pelinovsky E.N., Ruvinskaya E.A. Higher-order (2+4) Korteweg-de Vries - like equation for interfacial waves in a symmetric three-layer fluid. *Physics Fluids*. 2011, vol. 23, 116602.

Kurkina O. E., Kurkin A.A., Ruvinshaya E.A., Pelinovsky E.N., Soomere T. Dynamics of solitons in a nonintegrable version of the modified Korteweg – de Vries equation. *JETP Letters*, 2012, vol. 95, No. 2, 91-95.

General Scheme for KdV-like equations for stratified flow

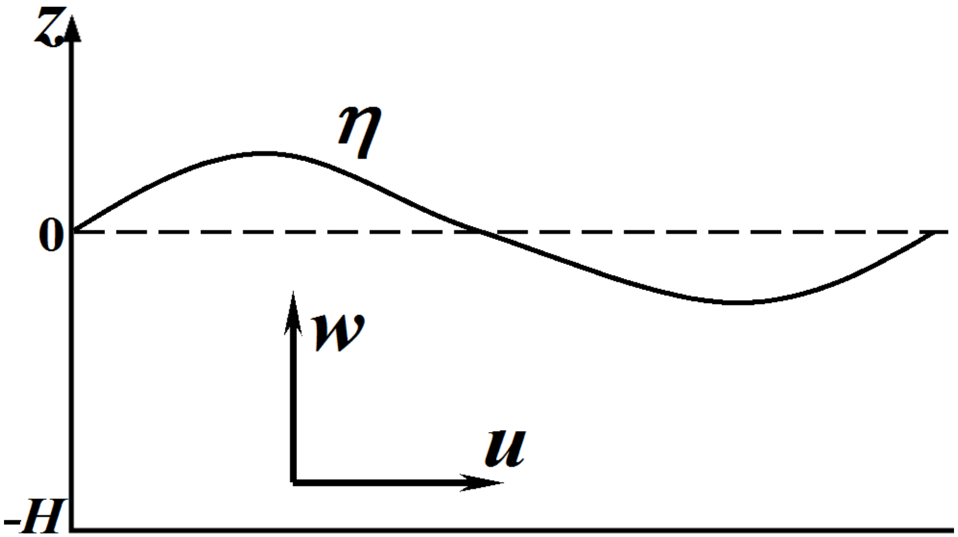
Euler Equations

$$\rho \frac{du}{dt} + \frac{\partial p}{\partial x} = 0$$

$$\rho \frac{dw}{dt} + \frac{\partial p}{\partial z} + \rho g = 0$$

$$div\vec{V} = 0$$

$$\frac{d\rho}{dt} = 0$$



Boundary conditions

w=0 for z=-H

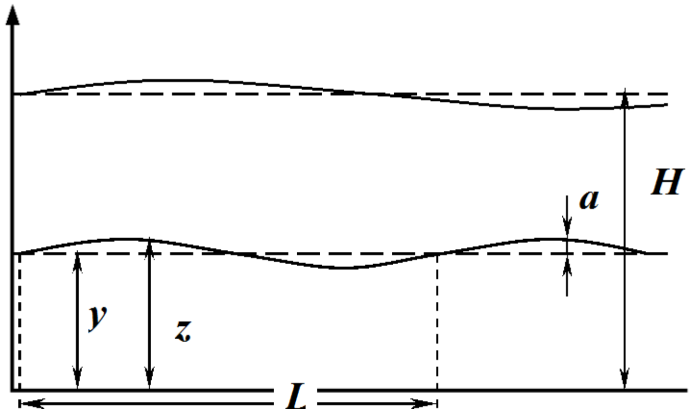
$$w = \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x}$$

p=0 for $z = \eta(x,t)$

Lagrange coordinate:

$$z \to y = z - \eta(x,z,t)$$

$$\rho(x,z,t) = \rho_0(z - \varsigma(x,z,t))$$



Asymptotic series:

$$\xi = X - cT$$

$$X = \mu x \qquad T = \mu t$$

$$\varepsilon \sim a/H$$

$$\mu \sim H/L$$

$$\tau_{ij} = \varepsilon^i \overline{\mu}^j T$$

$i,j = 0,1,2, \ldots, i + j > 0$

$$\frac{\partial}{\partial T} = -c\frac{\partial}{\partial \xi} + \sum_{i,j=0} \varepsilon^i \overline{\mu}^j \frac{\partial}{\partial \tau_{ij}}$$

$$\frac{\partial}{\partial X} = \frac{\partial}{\partial \xi}$$

$\overline{\mu} = \mu^2$ - dispersion

ε - nonlinearity

$$\eta(\xi,y,\tau) = \varepsilon\left(A(\xi,\tau)\Phi(y) + \sum_{i,j}\varepsilon^i \overline{\mu}^j \eta_{ij}(\xi,y,\tau)\right)$$

$$u(\xi,y,\tau) = U(y) + \varepsilon\sum_{i,j}\varepsilon^i \overline{\mu}^j u_{ij}(\xi,y,\tau)$$

Grimshaw R., Pelinovsky E., Poloukhina O., 2002: Nonlinear Processes In Geophysics

Inhomogeneous Eigenvalue Problems

$$\frac{\partial}{\partial y}\left\{\rho_0 (c-U)^2 \frac{\partial^2 \eta_{ij}}{\partial \xi \partial y}\right\} + \rho_0 N^2 \frac{\partial \eta_{ij}}{\partial \xi} = M_{ij}$$

$$\eta_{ij} = 0 \quad \text{for} \quad y = -H$$

$$\frac{\partial \eta_{ij}}{\partial \xi} = R_{ij} \quad \text{for} \quad y = 0$$

$$N^2(y) = -\frac{g}{\rho_0(y)} \frac{d\rho_0}{dy}$$

Brent-Vaisala Frequency

$\varepsilon^0, \overline{\mu}^0$:

$$L\Phi \equiv \frac{d}{dy}\left[\rho_0 (c-U(y))^2 \frac{d\Phi}{dy}\right] + \rho_0 N^2(y)\Phi = 0$$

$$\Phi(y=-H) = 0$$

$$\Phi(y=0) = \sigma (c-U)^2 \frac{d\Phi}{dy}$$

Vertical displacement:

$$\eta(\xi, y, \tau) = \varepsilon\left(A(\xi,\tau)\Phi(y) + \varepsilon A^2 T_n(y) + \overline{\mu} A_{\xi\xi} T_d(y) + \ldots\right)$$

-Linear Mode

$$L\Phi \equiv \frac{d}{dy}\left[\rho_0\left(c - U(y)\right)^2 \frac{d\Phi}{dy}\right] + \rho_0 N^2(y)\Phi = 0$$

$$\Phi(y = -H) = 0$$

$$\Phi(y = 0) = \sigma(c - U)^2 \frac{d\Phi}{dy}$$

-Nonlinear Correction

$$LT_n = -\alpha \frac{d}{dy}\left[\rho_0(c - U)\frac{d\Phi}{dy}\right] + \frac{3}{2}\frac{d}{dy}\left[\rho_0(c - U)^2\left(\frac{d\Phi}{dy}\right)^2\right]$$

-Dispersive Correction

$$LT_d = -2\beta \frac{d}{dy}\left[\rho_0(c - U)\frac{d\Phi}{dy}\right] - \rho_0(c - U)^2 \Phi$$

KdV $A_T + cA_X + \varepsilon\alpha AA_X + \overline{\mu}\beta A_{XXX} = 0$

Benney, 1966

KdV2
$$A_T + cA_X + \varepsilon\alpha AA_X + \overline{\mu}\beta A_{XXX} +$$
$$+ \varepsilon^2\alpha_1 A^2 A_X + \overline{\mu}^2\beta_1 A_{XXXXX} + \varepsilon\overline{\mu}\left(\gamma_1 AA_{XXX} + \gamma_2 A_X A_{XX}\right) = 0$$

KdV3

Koop & Butler, 1981; Lamb & Yan, 1996

$$A_T + cA_X + \varepsilon\alpha AA_X + \overline{\mu}\beta A_{XXX} +$$
$$+ \varepsilon^2\alpha_1 A^2 A_X + \overline{\mu}^2\beta_1 A_{XXXXX} + \varepsilon\overline{\mu}\left(\gamma_1 AA_{XXX} + \gamma_2 A_X A_{XX}\right) +$$
$$+ \varepsilon^3\alpha_2 A^3 A_X + \overline{\mu}^3\beta_2 A_{7X} +$$
$$+ \varepsilon\overline{\mu}^2\left(\gamma_{21} A_{XX} A_{XXX} + \gamma_{22} A_X A_{XXXX} + \gamma_{23} AA_{XXXXX}\right) +$$
$$+ \varepsilon^2\overline{\mu}\left(\gamma_{31} A_X^{\ 3} + \gamma_{32} AA_X A_{XX} + \gamma_{33} A^2 A_{XXX}\right) = 0$$

Korteweg-de Vries equation:

$\varepsilon \sim \overline{\mu}$

COEFFICIENTS

$$\alpha I = 3\int_{-H}^{0} \rho_0 (c-U)^2 (d\Phi/dy)^3 dy \qquad \beta I = \int_{-H}^{0} \rho_0 (c-U)^2 \Phi^2 dy$$

$$\alpha_1 I = \int_{-H}^{0} \rho_0 dy \left\{ 3(c-U)^2 \left[3(dT_n/dy) - 2(d\Phi/dy)^2 \right] (d\Phi/dy)^2 + \alpha(c-U) \left[5(d\Phi/dy)^2 - 4(dT_n/dy) \right] (d\Phi/dy) - \alpha^2 (d\Phi/dy)^2 \right\}$$

$$\beta_1 I = \int_{-H}^{0} \rho_0 dy \left\{ 2\beta(c-U) \left[\Phi^2 - (d\Phi/dy)(dT_d/dy) \right] - \beta^2 (d\Phi/dy)^2 + (c-U)^2 \Phi T_d \right\}$$

$$\gamma_1 I = -\int_{-H}^{0} \rho_0 dy \left\{ 2(c-U) \left[\alpha(dT_d/dy) + 2\beta(dT_n/dy) \right] (d\Phi/dy) + 2\alpha\beta(d\Phi/dy)^2 - 2\alpha(c-U)\Phi^2 + (c-U)^2 \Phi^2 (d\Phi/dy) - \right.$$
$$\left. - 4\beta(c-U)(d\Phi/dy)^3 - (c-U)^2 \left[3(dT_d/dy)(d\Phi/dy)^2 + 2T_n\Phi \right] \right\}$$

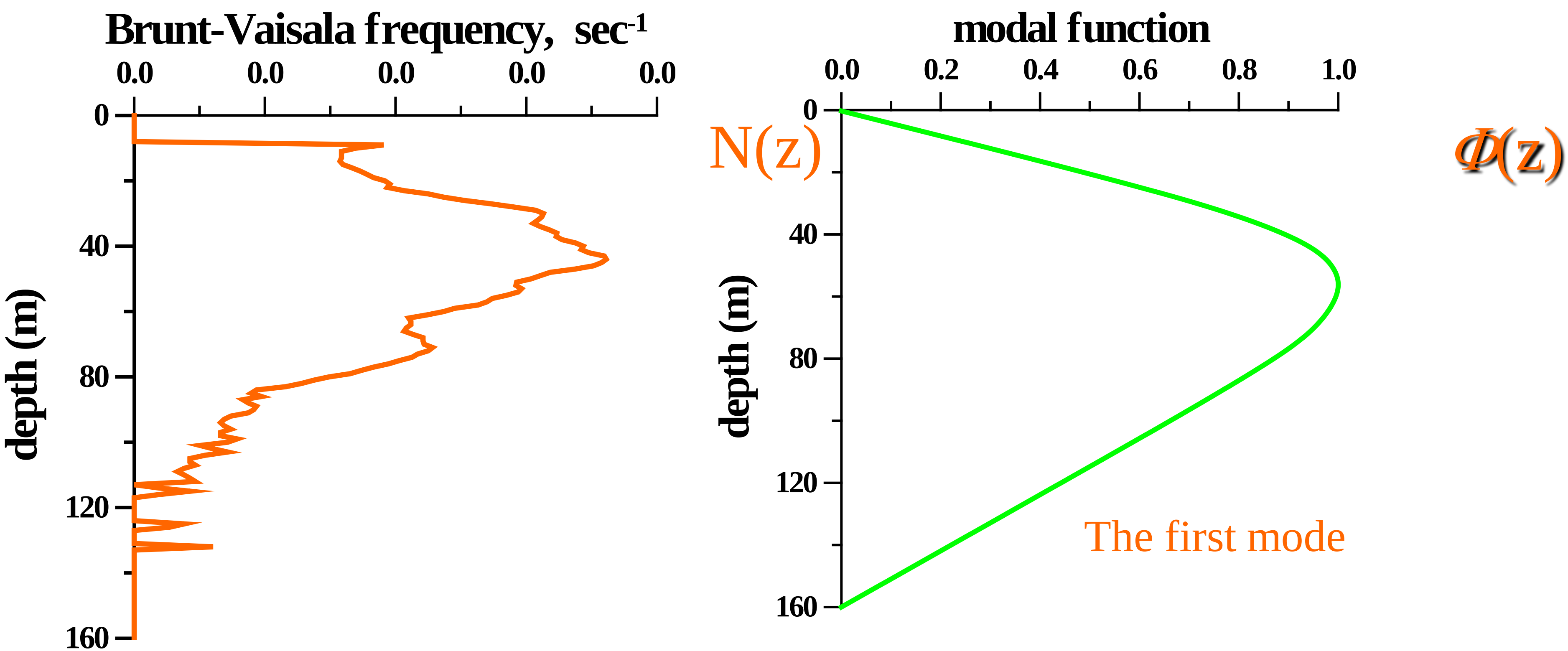
$$I = \int_{-H}^{0} \rho_0 dy \left\{ (c-U) \left[2\beta(d\Phi/dy)^3 + 6\alpha\Phi^2 \right] - 3\alpha\beta(d\Phi/dy)^2 - 2(c-U)^2 \left[\Phi^2 (d\Phi/dy) - 3T_n\Phi \right] - \right.$$
$$\left. - 6\alpha(c-U)(dT_d/dy)(d\Phi/dy) + 3(c-U)^2 dT_d/dy)(d\Phi/dy)^2 \right\}$$

$$I = 2\int_{-H}^{0} \rho_0 (c-U)(d\Phi/dy)^2 dy$$

$U(y)$ - horizontal shear stable flow

$\rho_0(y)$ - density stratification

MODAL STRUCTURE



Eigenvalue problem for Φ and c

$$\frac{d}{dz}\left[(c-U(z))^2\frac{d\Phi}{dz}\right]+N(z)^2\Phi=0,$$

$$\Phi(0)=\Phi(H)=0$$

$$\Phi_{\max}=1$$

Nonlinear Correction to Mode Structure

$$\frac{d}{dz}\left[(c-U)^2\frac{dT}{dz}\right]+N^2T=$$

$$=\frac{3}{2}\frac{d}{dz}\left[(c-U)^2\left(\frac{d\Phi}{dz}\right)^2\right]-\alpha\frac{d}{dz}\left[(c-U)\frac{d\Phi}{dz}\right]$$

T = 0 *where* z = 0, H
T = 0 *where* Φ(z) = 1

$$\zeta(z,x,t)=\eta(x,t)\Phi(z)+\eta^2T(z)$$

GARDNER EQUATION

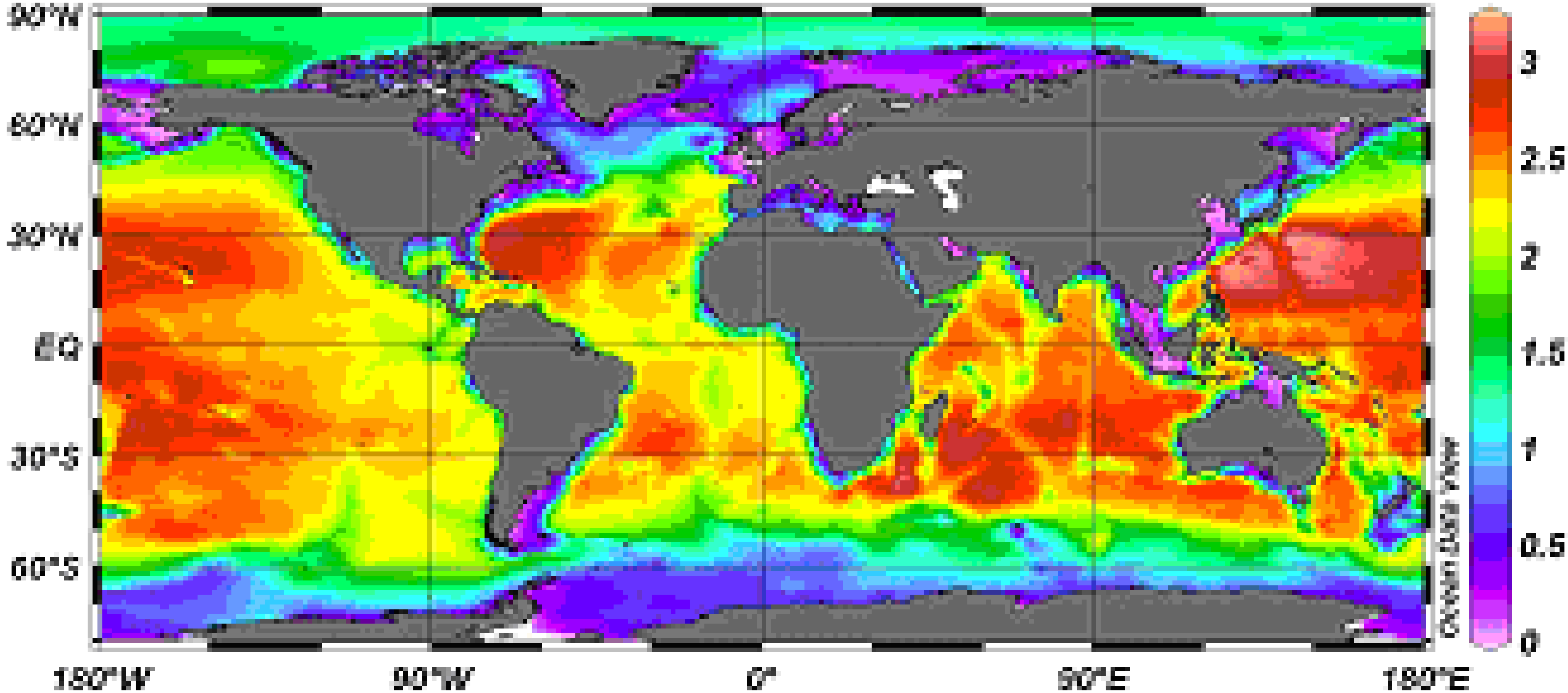
$$\frac{\partial\eta}{\partial t}+(c+\alpha\eta+\alpha_1\eta^2)\frac{\partial\eta}{\partial x}+\beta\frac{\partial^3\eta}{\partial x^3}=0$$

$$\alpha_1=-\frac{3}{2}\frac{\int\Theta dz}{\int(c-U)(d\Phi/dz)^2dz}$$

$$\Theta=(c-U)^2(d\Phi/dz)^2\left[2(d\Phi/dz)^2-3(dT/dz)\right]-$$

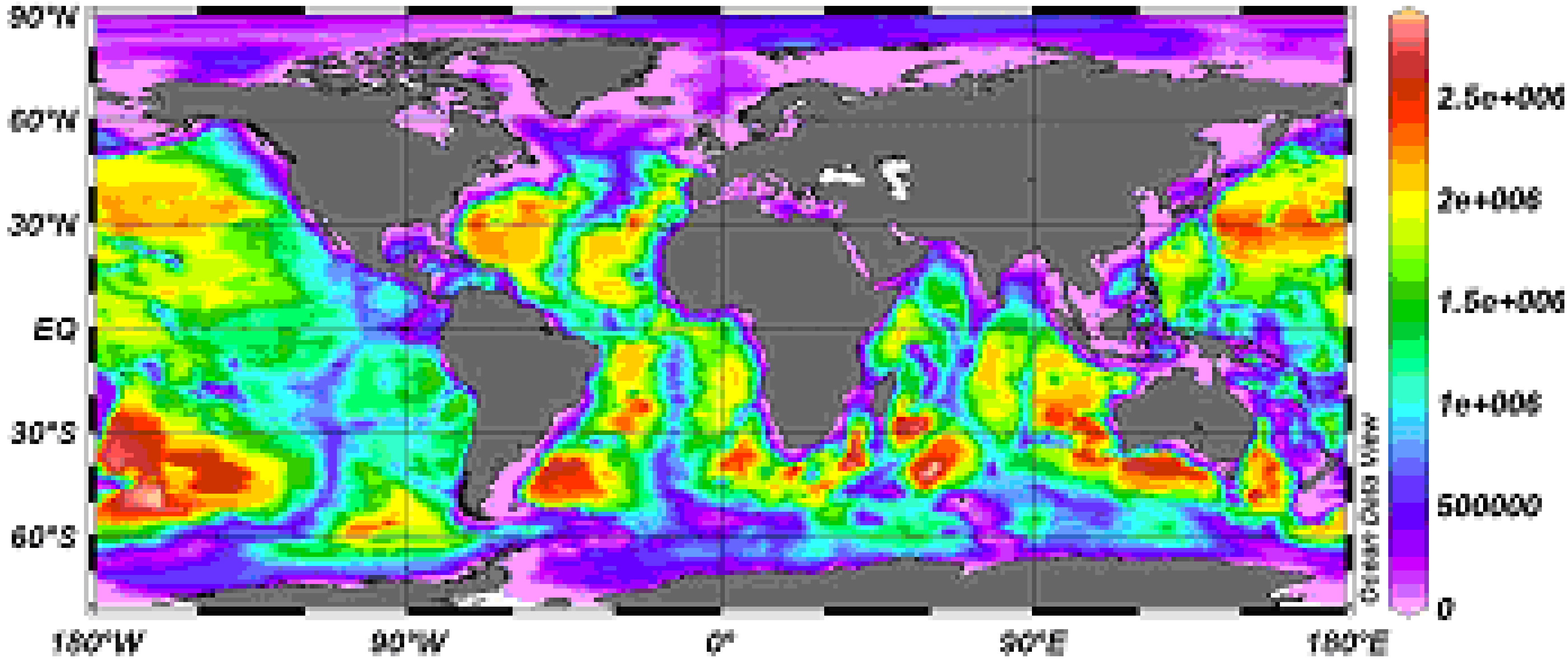
$$-\frac{\alpha}{3}(c-U)(d\Phi/dz)\left[5(d\Phi/dz)^2-4(dT/dz)\right]+\frac{\alpha^2}{3}(d\Phi/dz)^2$$

Cubic nonlinear coefficient for stratified water may be both signs

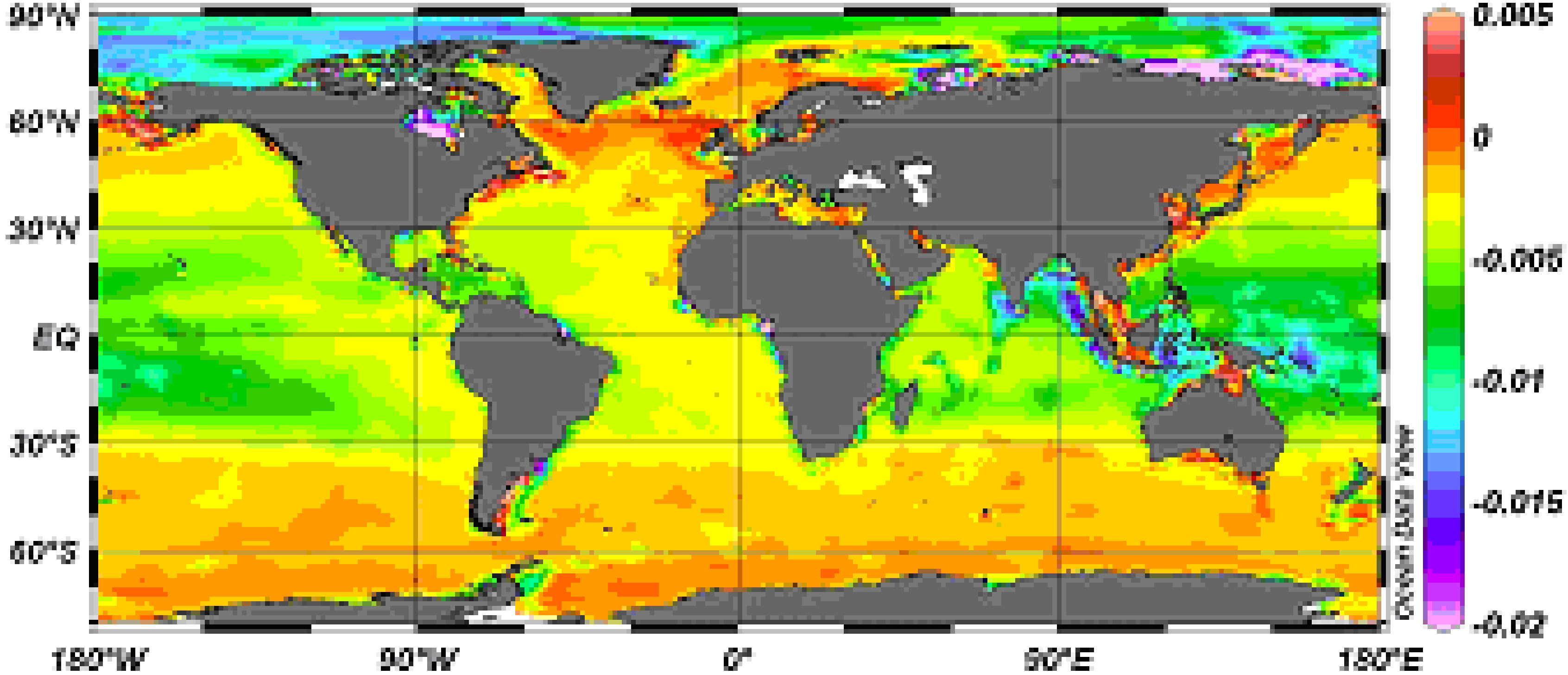


Linear Long Internal Wave Speed, c

Grimshaw, R., **Pelinovsky, E., and Talipova, T.** Modeling internal solitary waves in the coastal ocean. *Survey in Geophysics*, 2007, vol. 28, No. 4, 273-298

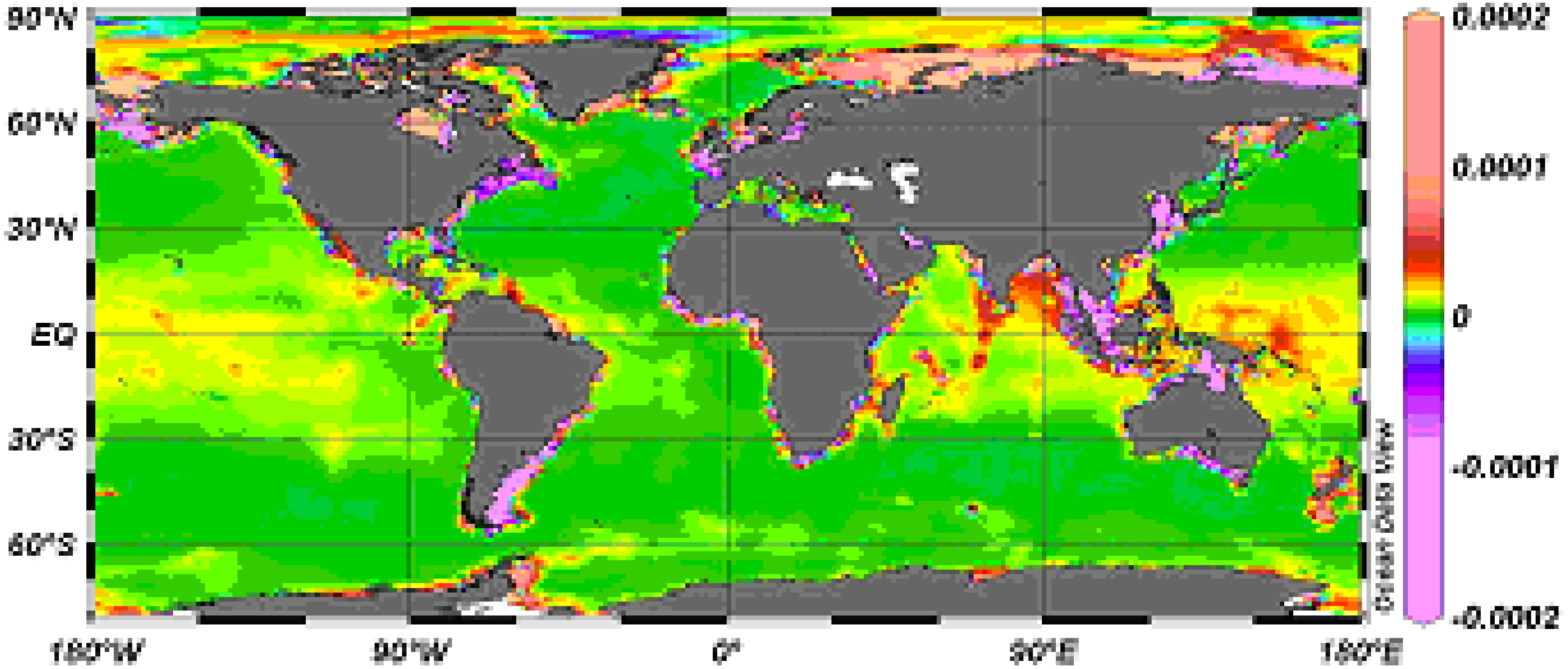


Dispersion Coefficient



Quadratic Nonlinear Term

Varied Sign!



Cubic Nonlinear Term

Varied Sign!

Gardner's Solitons

sign of α_1

$$u(x,t) = \frac{A}{1 + B\cosh(\gamma(x - Vt))},$$

$$A = \frac{6\beta\gamma^2}{\alpha},$$

$$B^2 = 1 + \frac{6\beta\alpha_1\gamma^2}{\alpha^2},$$

$$V = \beta\gamma^2$$

$$a = \frac{A}{1+B}$$

$\alpha_1 < 0$

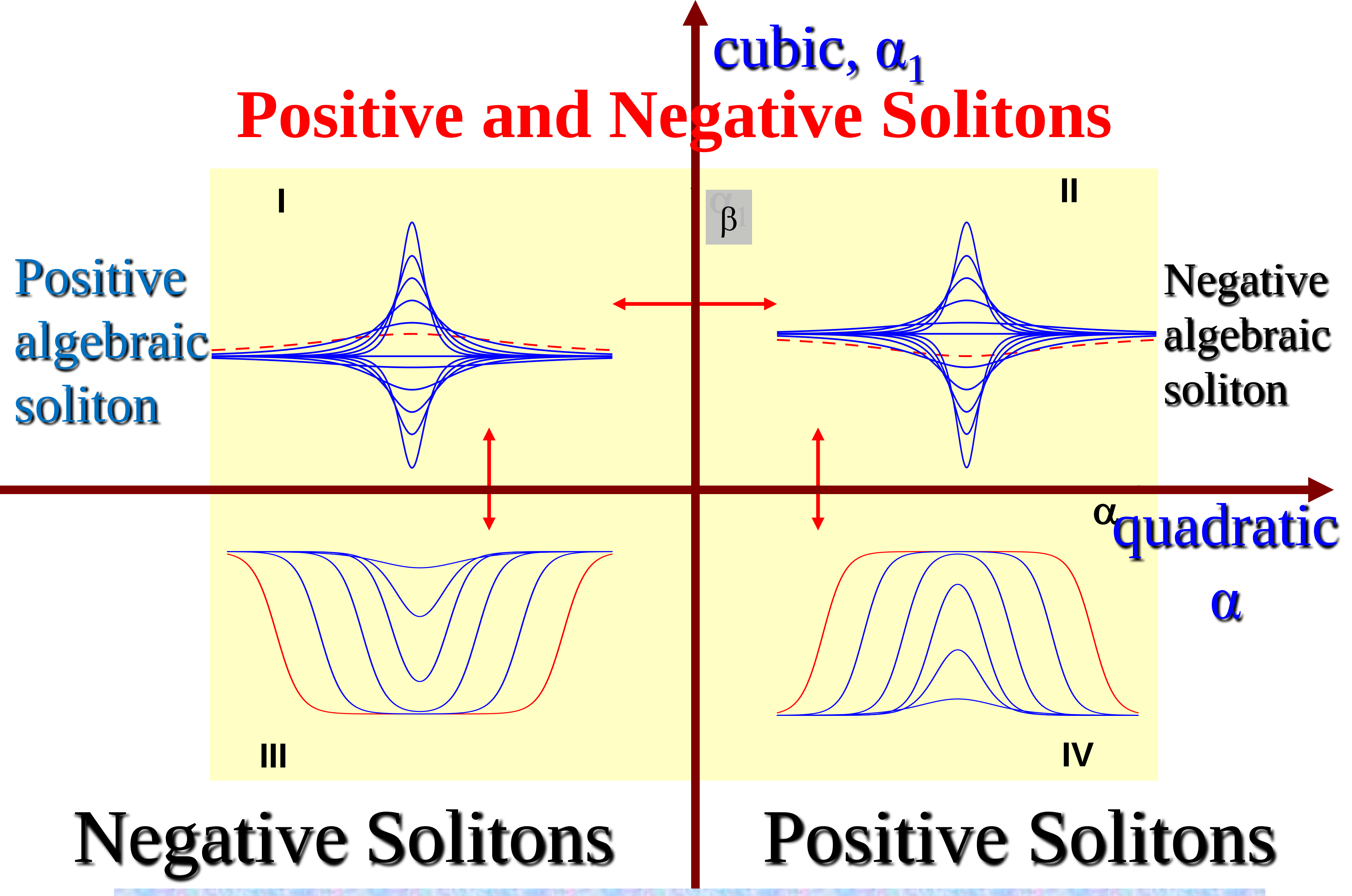
Limited amplitude

$a_{\lim} = -\alpha / \alpha_1$

$\alpha_1 > 0$

Two branches of solitons of both polarities, algebraic soliton $a_{\lim} = -2\alpha / \alpha_1$

Positive and Negative Solitons



Sign of the cubic term is principal!

N – Soliton Solutions

JOURNAL OF EXPERIMENTAL AND THEORETICAL PHYSICS VOLUME 89, NUMBER 1 JULY 1999

Dynamics of large-amplitude solitons

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E. N. Pelinovskiĭ[†)]

Cauchy Problem $(\alpha_1 < 0)$

CHAOS VOLUME 12, NUMBER 4 DECEMBER 2002

Generation of large-amplitude solitons in the extended Korteweg–de Vries equation

Roger Grimshaw[a)]
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Laboratory of Hydrophysics and Nonlinear Acoustics, Institute of Applied Physics, Nizhny Novgorod, Russia

N – Solitons and Cauchy Problem

positive cubic term

CHAOS VOLUME 10, NUMBER 2 JUNE 2000

On the generation of solitons and breathers in the modified Korteweg–de Vries equation

Simon Clarke, Roger Grimshaw, and Peter Miller
Department of Mathematics and Statistics, Monash University, Clayton, Victoria, Australia

Efim Pelinovsky and Tatiana Talipova
Institute of Applied Physics and Nizhny Novgorod Technical University, Nizhny Novgorod, Russia

Journal of Experimental and Theoretical Physics, Vol. 92, No. 3, 2001, pp. 529–534.
Translated from Zhurnal Éksperimental'noĭ i Teoreticheskoĭ Fiziki, Vol. 119, No. 3, 2001, pp. 606–612.

MISCELLANEOUS

Dynamics of Localized Waves with Large Amplitude in a Weakly Dispersive Medium with a Quadratic and Positive Cubic Nonlinearity

A. V. Slyunyaev

Grimshaw, R., Slunyaev, A., and Pelinovsky, E. Generation of solitons and breathers in the extended Korteweg-de Vries equation with positive cubic nonlinearity, *Chaos*, 2010, vol. 20, 013102

Again High-Order KdV Equation for stratified flow

For steady-state waves moved with constant speed, c the Euler equations can be reduced to 2D

Dubreil-Jacotin-Long equation

$$\Delta\eta + \frac{N^2(z-\eta)}{c^2}\eta = 0$$

Boundary conditions $\eta = 0$ at z=0, z = H and infinity x

Remain $$N^2(z) = -\frac{g}{\rho_0(z)}\frac{d\rho_0}{dz}$$ Brent-Vaiasala Frequency

If N = const, a wave of any amplitude is linear!

$$\Delta\eta + \frac{N^2(z-\eta)}{c^2}\eta = 0$$

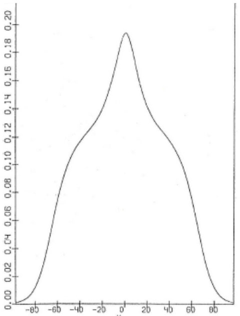
If N = const, a wave of any amplitude is linear!

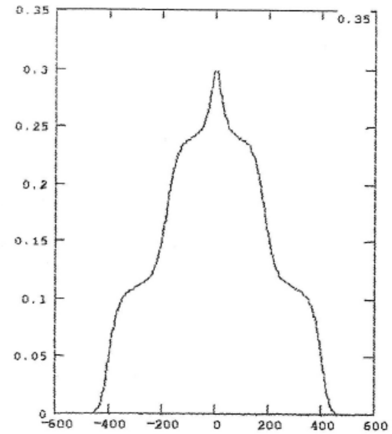
If stratification is weak $N^2(z) = N_0^2\left[1+\varepsilon P_m(z)\right]$

P is polynomial, all terms are the same order

As a result $$\frac{d^2\eta}{dx^2} - c\eta + P_m(\eta) = 0$$

Multi-humped solitary waves





M. Dunphy, C. Subich, and M. Stastna. Spectral methods for internal waves: indistinguishable density profiles and double-humped solitary waves. ***Nonlinear Processes in Geophysics*****, 2011,** vol. **18, 351–358**

Derzho O. Multi-scaled solitary waves. ***Nonlinear Processes in Geophysics*****, 2017, Discussion**

Elastic Waves in Bimodular Media

$$\rho \frac{\partial^2 \varepsilon}{\partial t^2} = \frac{\partial^2 \sigma}{\partial x^2} + \gamma \frac{\partial^4 \sigma}{\partial x^4}, \qquad \varepsilon = \frac{1}{E}\left(\sigma + g\left|\sigma\right|\right).$$

σ – stress

ε - deformation

different response to tensile and compressive stresses

$$\frac{\partial u}{\partial t} = \frac{\partial |u|}{\partial x} + \frac{\partial^3 u}{\partial x^3}$$

"Modular" KdV Equation

Rudenko O.V. Modular solitons, *Doklady Mathematics*, 2016, vol. 94, 708-711

Nazarov V., Kiashko S., Radostin A. Wave processes in bimodular media. *Radiophysics and Quantum Electronics*, 2016, vol. 59, 275-285

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Cnoidal waves in solids

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A R T I C L E   I N F O

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Dedicated to Professor Ioannis Vardoulakis1

A B S T R A C T

Cnoidal waves are nonlinear and exact periodic stationary waves, well known in the shallow water theory of fluid mechanics. In this study we retrieve such periodic stationary wave solutions as singularities of the problem of homogeneous volumetric deformation of a rate-dependent, heterogeneous solid material. In accordance to the classical Hill stationary wave localization instability, which provides velocity gradient discontinuities in shear failure, cnoidal waves are dilational and compactional manifestations of volumetric localization along lines of stress discontinuities. They therefore emerge along the volumetric component of the classical slip line field theory, with their regular distance being a tell tale indication of rate-dependent volumetric deformation. We discuss applications for the dominant mode of I1 compaction in geomaterials where distinct cnoidal wave instabilities appear as localisation features in compaction. We also discuss the case of localisation features in a classical (J2 plastic) material where a small but important cnoidal contribution may trigger equidistant bands of localisation known as Lüders lines. We therefore postulate that cnoidal waves constitute fundamental material instabilities stemming from the propagation of elasto-plastic P-waves.

Plasma Waves and KdV-like equations

Schamel, H. A modified Korteweg-de Vries equation for ion acoustic waves due to resonant electrons. *J. Plasma Phys*. 9, 377–387 (1973)

$$\frac{\partial u}{\partial t} + \frac{5}{2}\frac{\partial |u|^{3/2}}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

Schamel Equation

Gardner equation

$$\frac{\partial u}{\partial t} + (\alpha u + \alpha_1 u^2)\frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0$$

Ruderman M.S., Talipova T., Pelinovsky, E. Dynamics of modulationally unstable ion-acoustic wave packets in plasmas with negative ions. *J. Plasma Physics,* 2008, vol. 74, No. 5, 639-656

S.A. El-Tantawy, E.I. El-Awady, R. Schlickeiser. Freak waves in a plasma having Cairns particles. *Astrophys Space Sci* 2015, vol. 360, 49.

S.A. El-Tantawy. Rogue waves in electronegative space plasmas: The link between the family of the KdV equations and the nonlinear Schrödinger equation. *Astrophys Space Sci*, 2016, vol. 361. 164

Logarithmic KdV equation

Granular chains with Hertzian interaction forces

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(u \log|u|\right) + \frac{\partial^3 u}{\partial x^3} = 0$$

Gaussian soliton

$$u(x-ct) = \exp\left[c + \frac{1}{2} - \frac{(x-ct-x_0)^2}{4}\right]$$

R. Carles and D. Pelinovsky. On the orbital stability of Gaussian solitary waves in the log-KdV equation, *Nonlinearity,* 2014, vol. 27, 3185 -3202.

E. Dumas and D.E. Pelinovsky. Justification of the log-KdV equation in granular chains: the case of precompression, *SIAM J. Math. Anal.*, 2014, vol. 46, 4075 -4103.

G. James and D. Pelinovsky. Gaussian solitary waves and compactons in Fermi-Pasta-Ulam lattices with Hertzian potentials, *Proc. Roy. Soc. A*, 2014, vol. 470, 20130465.

Summary of "physical" KdV-like models

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}f(u)+\frac{\partial^3 u}{\partial x^3}=0$$

$$f(u)=u^m$$

m = 2 (KdV), 3 (mKdV), 5

$$f(u)=|u|^m$$

m = 1 (modular), 3/2 (Schamel)

$$f(u)=u\log|u|$$

Gaussian solitons

$$f(u)=\alpha u^2+\alpha_1 u^3$$

Gardner equation

$$f(u)=\alpha u^2+\alpha_1 u^3+\alpha_2 u^4+\alpha_3 u^5+\dots$$

Multi-humped solitary waves

But Instabilities.......

$$\frac{\partial u}{\partial t} + u^m \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

m > 0 – any integer

Soliton

$$u = \left\{ \frac{(m+1)(m+2)}{2} c \bullet \mathrm{sech}^2 \left[\frac{\mathrm{m}\sqrt{\mathrm{c}}}{2} (x - ct) \right] \right\}^{1/m}$$

Mass

$$M = \int u dx \sim c^{\frac{1}{m} - \frac{1}{2}}$$

m < 2 (m > 2) **M increases (decreases) with amplitude, c**

m= 2 (mKdV) **M does not depend from c**

But Instabilities…….

$$\frac{\partial u}{\partial t} + u^m \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

m > 0 – any integer

Soliton

$$u \sim \left\{ c \bullet \mathrm{sech}^2 \left[\frac{\mathrm{m}\sqrt{\mathrm{c}}}{2} (x - ct) \right] \right\}^{1/m}$$

Momentum (Energy)

$$E = \int u^2 dx \sim c^{\frac{2}{m} - \frac{1}{2}}$$

m < 4 (m > 4) **E increases (decreases) with amplitude, c**

m= 4 **M does not depend from c**

Solitons in critical and supercritical cases are unstable and blow-up

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Numerical study of blow-up and dispersive shocks in solutions to generalized Korteweg–de Vries equations

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HIGHLIGHTS

- Numerical study of soliton stability in critical and supercritical generalized KdV equations.
- Numerical identification of the blow-up mechanism.
- Numerical study of the small dispersion limit and the ϵ-dependence of the blow-up time.

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ABSTRACT

We present a detailed numerical study of solutions to general Korteweg–de Vries equations with critical and supercritical nonlinearity, both in the context of dispersive shocks and blow-up. We study the stability of solitons and show that they are unstable against being radiated away and blow-up. In the L_2 critical case, the blow-up mechanism by Martel, Merle and Raphaël can be numerically identified. In the limit of small dispersion, it is shown that a dispersive shock always appears before an eventual blow-up. In the latter case, always the first soliton to appear will blow up. It is shown that the same type of blow-up as for the perturbations of the soliton can be observed which indicates that the theory by Martel, Merle and Raphaël is also applicable to initial data with a mass much larger than the soliton mass. We study the scaling of the blow-up time t^* in dependence of the small dispersion parameter ϵ and find an exponential dependence $t^*(\epsilon)$ and that there is a minimal blow-up time t_0^* greater than the critical time of the corresponding Hopf solution for $\epsilon \to 0$. To study the cases with blow-up in detail, we apply the first dynamic rescaling for generalized Korteweg–de Vries equations. This allows to identify the type of the singularity.

$$\frac{\partial u}{\partial t} - u^m \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

In our 3-layer flow a sign “minus” if m > 3

Modulational Instability of weakly nonlinear group

$$\frac{\partial u}{\partial t} + \varepsilon(<<1)\frac{\partial}{\partial x} f(u) + \frac{\partial^3 u}{\partial x^3} = 0$$

Asymptotic series

$$u(\theta) = A(\varepsilon x, \varepsilon t)\cos\theta + \varepsilon u_1(\theta, \varepsilon x, \varepsilon t) + \varepsilon^2 \ldots$$

$$\theta = \omega_0 t - k_0 x + \varphi(\varepsilon x, \varepsilon t) \qquad \omega_0 = -k_0^3$$

Deriving NLS

$$i\frac{\partial A}{\partial t} + \frac{\partial^2 A}{\partial x^2} + s|A|^2 A = 0$$

s = - 1 stable

s = + 1 unstable

KdV weakly nonlinear group

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

Any sign of nonlinearity

V.E. Zakharov, E.A. Kuznetsov, Multi-scale expansions in the theory of systems integrable by the inverse scattering transform, *Physica D*, 1986, vol. 18, 455–463.

Happy Birthday, Eugene!

NLS

$$i\frac{\partial A}{\partial t} + \frac{\partial^2 A}{\partial x^2} - |A|^2 A = 0$$

Sign “minus” for any sign of nonlinearity

KdV groups are stable!

Frontiers in Nonlinear Physics, Volga River, 2004

mKdV weakly nonlinear group

$$\frac{\partial u}{\partial t} + \alpha_1 u^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

Grimshaw R., Pelinovsky E., Talipova, T., Ruderman M. and Erdelyi R. Short-lived large-amplitude pulses in the nonlinear long-wave model described by the modified Korteweg–de Vries equation. *Studied Applied Mathematics*, 2005, vol. 114, No. 2, 189-210

NLS

$$i\frac{\partial A}{\partial t} + \frac{\partial^2 A}{\partial x^2} + \mathrm{sgn}(\alpha_1)\,|A|^2\,A = 0$$

mKdV groups are unstable if $\alpha_1 > 0$

Gardner weakly nonlinear group

$$\frac{\partial u}{\partial t}+\left(\alpha u+\alpha_1 u^2\right)\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}=0$$

NLS

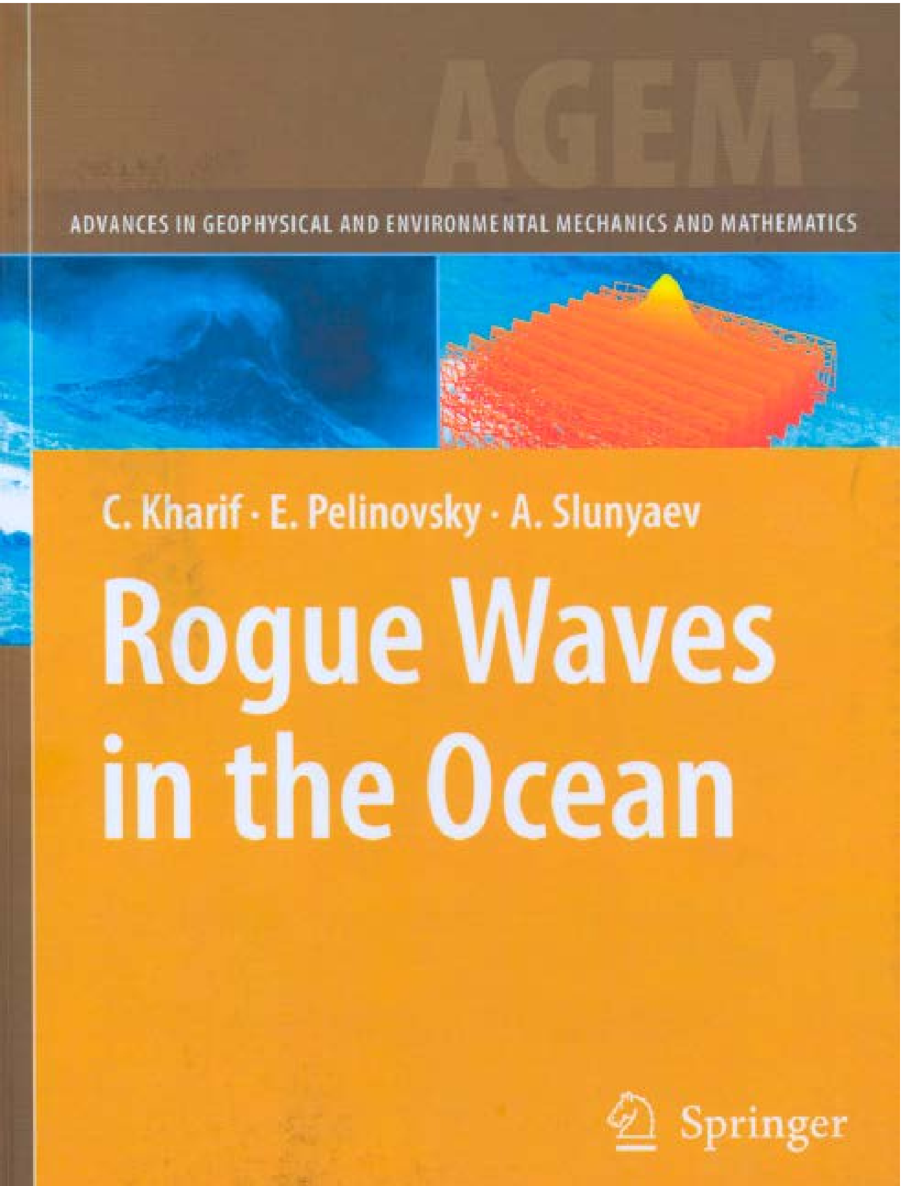
$$i\frac{\partial A}{\partial t}+\frac{\partial^2 A}{\partial x^2}+\mathrm{sgn}\left(\alpha_1-\frac{\alpha^2}{6k_0^2}\right)|A|^2 A=0$$

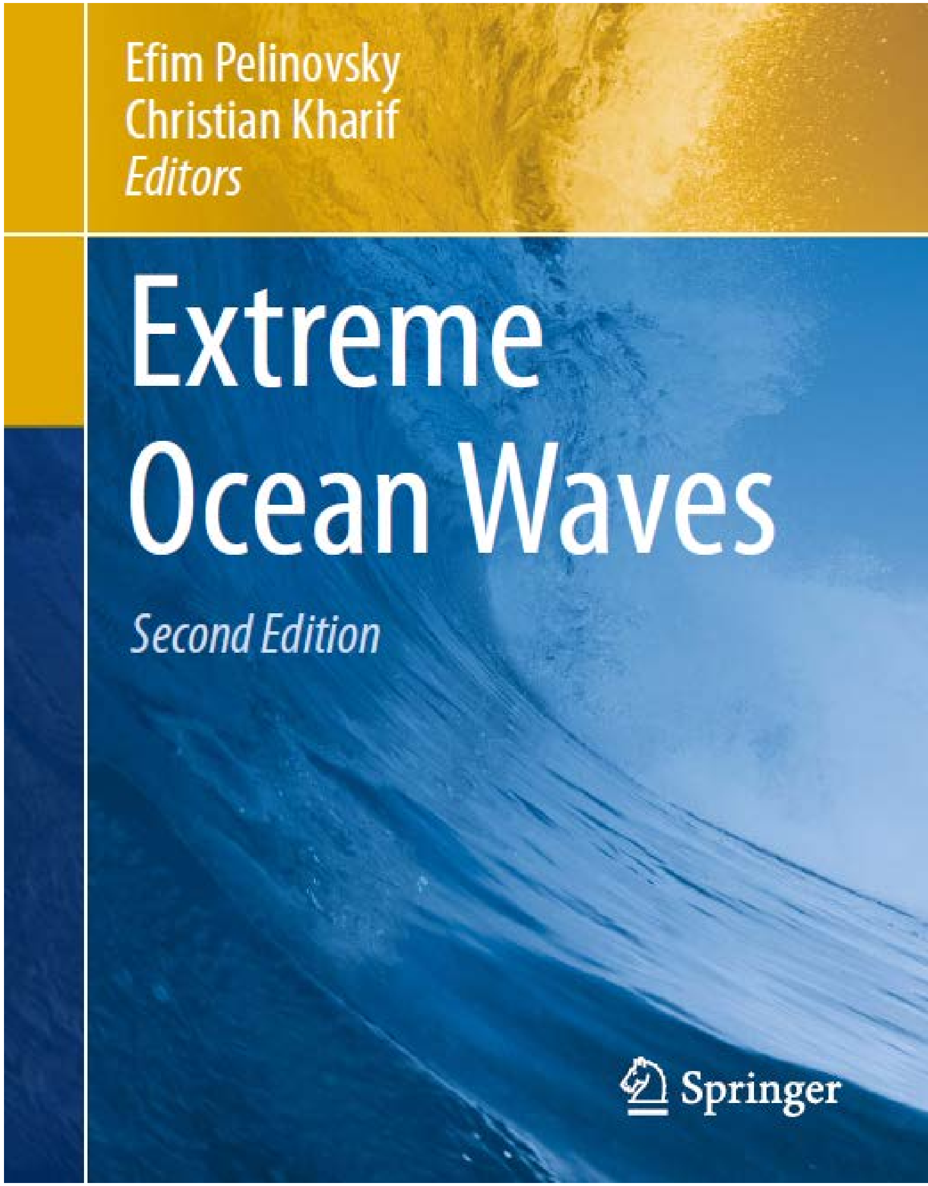
Gardner groups are unstable if α_1 > 0 and k > k_{cr}

Gardner breather at α_1 > 0

$$u=2\frac{\partial}{\partial x}\mathrm{atan}\frac{l\cosh(\Psi)\cos(\theta)-k\cos(\Phi)\sinh(\kappa)}{l\sinh(\Psi)\sin(\theta)+k\sin(\Phi)\cosh(\kappa)}$$

Rogue Waves in KdV-like systems due to modulational instability, soliton focusing and dispersive focusing





High-order KdV weakly nonlinear group

$$\frac{\partial u}{\partial t} + su^{2m}\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

$$s = \pm 1$$

High-order NLS

$$i\frac{\partial A}{\partial t} + \frac{\partial^2 A}{\partial x^2} + sB_m |A|^{2m} A = 0$$

$$B_m = \frac{(2m+1)!}{2^{2m}(2m+1)m!(m+1)!}$$

E. Tobish (Kartashova) and E. Pelinovsky in preparation

Wave packets are unstable!

Increment

$$\Gamma = \sqrt{3A_0}K\sqrt{2mB_m A_0^{2m} - 3K^2}$$

Envelope Soliton

$$A(x,t) = A_0 \frac{\exp\left[\dfrac{iB_m A_0^{2m} t}{m+2}\right]}{\cosh^{1/m}\left[n\sqrt{\dfrac{B_m}{3(m+2)}} A_0^m x\right]}$$

Soliton Energy

$$E = \int_{-\infty}^{+\infty} |A|^2 \, dx \sim A_0^{2-m}$$

Critical regime m = 2 is the same for KdV and NLS equations

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Collision of N-solitons in a fifth-order nonlinear Schrödinger equation

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HIGHLIGHTS

- N-solitons in a fifth-order nonlinear Schrödinger equation are presented.
- Developed bilinear forms via minimal use of auxiliary functions.
- Analyzed of gain and loss of amplitudes phenomena.
- Investigated the elastic and non-elastic collisions.

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ABSTRACT

The existence of N-solitons in a fifth-order nonlinear quintic Schrödinger equation is investigated through the use of a Hirota's differential operator, which utilizes only one auxiliary function. This minimal use of auxiliary functions is a novel modification of the bilinear forms that allow us to obtain a larger and a more general class of N-solitons. As the number of auxiliary functions increases, the obtained solutions are more restricted and may not exhibit some of the important behaviors, thus the novel solutions given here are more general. Several classes of 4-solitons exhibiting elastic collisions, and non-elastic collisions that lead to the gain and the loss of amplitudes after collision in a conservative system are presented. Various plots to support the analytic results are presented with propagations illustrated along the x-axis. The explicit expressions for the solitons turn out to be the same as those given by Hirota for the third-order case, but the dispersion relation is different.

Summary of "physical" KdV-like models

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) + \frac{\partial^3 u}{\partial x^3} = 0$$

$$f(u) = u^m$$

m = 2 (KdV), 3 (mKdV), 5

$$f(u) = |u|^m$$

m = 1 (modular), 3/2 (Schamel)

$$f(u) = u \log|u|$$

Gaussian solitons

$$f(u) = \alpha u^2 + \alpha_1 u^3$$

Gardner equation

$$f(u) = \alpha u^2 + \alpha_1 u^3 + \alpha_2 u^4 + \alpha_3 u^5 + \ldots$$

Multi-humped solitary waves