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The dynamics of quantum vortices in a quasi-two-dimensional Bose-Einstein condensate with two "holes"

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1. INTRODUCTION

We consider a trapped quasi-2D Bose-Einstein condensate in the Thomas-Fermi regime, with a spatially nonuniform equilibrium 2D density profile $\rho(\mathbf{x})$, and use hydrodynamic "anelastic" approximation to study slow dynamics of quantum "point" vortices, each with circulation $\Gamma = 2\pi\hbar/m_{\mathrm{atom}}$:

$$\Gamma\sigma_n\rho(\mathbf{x}_n)\dot{x}_n = \frac{\partial H}{\partial y_n}, \qquad -\Gamma\sigma_n\rho(\mathbf{x}_n)\dot{y}_n = \frac{\partial H}{\partial x_n}, \qquad \sigma_n = \pm 1. \tag{1}$$

Stream function (flux potential) for the divergence-free field $\rho\mathbf{V}$:

$$\rho V_x = \partial_y\psi, \qquad \rho V_y = -\partial_x\psi. \tag{2}$$

Vorticity:

$$\partial_x v_y - \partial_y v_x = -\nabla_{\mathbf{x}}\cdot\frac{1}{\rho(\mathbf{x})}\nabla_{\mathbf{x}}\psi(\mathbf{x}) = \omega(\mathbf{x}) \approx \Gamma\sum_n \sigma_n\delta(\mathbf{x}-\mathbf{x}_n). \tag{3}$$

Green's function:

$$-\nabla_{\mathbf{x}}\cdot\frac{1}{\rho(\mathbf{x})}\nabla_{\mathbf{x}}G(\mathbf{x},\mathbf{x}_0) = 2\pi\delta(\mathbf{x}-\mathbf{x}_0). \tag{4}$$

The Hamiltonian (if the density does not vanish anywhere):

$$H = \frac{1}{2}\int\frac{(\nabla_{\mathbf{x}}\psi)^2}{\rho(\mathbf{x})}d^2\mathbf{x} = \frac{1}{2}\int\psi\omega d^2\mathbf{x} = \frac{\Gamma^2}{4\pi}\sum_n F(\mathbf{x}_n) + \frac{\Gamma^2}{4\pi}{\sum_{m,n}}'\sigma_n\sigma_m G(\mathbf{x}_n,\mathbf{x}_m). \tag{5}$$

$$F(\mathbf{x}_n) = G(\mathbf{x}_n, \mathbf{x}_n + \mathbf{e}\xi(\mathbf{x}_n)), \qquad \xi(\mathbf{x}) = \xi_*[\rho_0/\rho(\mathbf{x})]^{1/2}, \quad \Lambda = \log(R_*/\xi_*) \gg 1. \tag{6}$$

The problem is to calculate Green's function. Substitution $G(\mathbf{x},\mathbf{x}_0) = \sqrt{\rho(\mathbf{x})\rho(\mathbf{x}_0)}g(\mathbf{x},\mathbf{x}_0)$ gives equation

$$[-\nabla^2_{\mathbf{x}} + \tilde{\kappa}^2(\mathbf{x})]g(\mathbf{x},\mathbf{x}_0) = 2\pi\delta(\mathbf{x}-\mathbf{x}_0), \tag{7}$$

with the coefficient

$$\tilde{\kappa}^2(\mathbf{x}) = \sqrt{\rho}\nabla^2_{\mathbf{x}}\frac{1}{\sqrt{\rho}}. \tag{8}$$

Only few explicit solutions are known so far. In particular, if $\tilde{\kappa}^2(\mathbf{x}) =$ const $= \kappa^2$, then

$$\rho_\kappa = \Big[\int_0^{2\pi} C(\varphi)\exp(\kappa x\cos\varphi + \kappa y\sin\varphi)\frac{d\varphi}{2\pi}\Big]^{-2}, \tag{9}$$

$$G_\kappa(\mathbf{x}_1,\mathbf{x}_2) = \sqrt{\rho_\kappa(\mathbf{x}_1)\rho_\kappa(\mathbf{x}_2)}K_0(\kappa|\mathbf{x}_1-\mathbf{x}_2|), \tag{10}$$

where $K_0(\ldots)$ is the corresponding modified Bessel function. In addition to this class of density profiles, the exact Green's function was found for the linear dependence ($\rho = x$ at $x > 0$ [J. R. Anglin, Phys. Rev. A **65**, 063611 (2002)]) and for the Gaussian density profile [V.R., JETP 124(6), (2017); arXiv:1612.00165v2].

In this work we consider one more, previously unexplored family of the dependences $\rho(\mathbf{x})$ for which the Green's function can be found exactly. In particular, in a certain region of parameters, it is a condensate forming an inhomogeneous planar disk with a hole (i.e., a ring). Since the density in this case vanishes not only at infinity, there is an opportunity of placing a few circulation quanta at each finite zero point and studying in detail their influence on the vortex motion.

In addition, the theory will be generalized to the case of a quasi-2D Bose-Einstein condensate which is not flat but is situated on a curved surface in the 3D space (for example, on a sphere, ellipsoid, or toroid). An analog of this situation in classical hydrodynamics is the motion of vortices in an ocean of variable depth on a spherical planet. In contrast to well studied dynamics of vortices in a homogeneous liquid layer on curved surfaces, a self-consistent theory of vortices in an inhomogeneous curved layer has not yet been developed.

2. MODEL OF A "DOUBLE-HOLE" CONDENSATE

Now we turn to a qualitatively more complicated situation than that described by Eqs.(9)-(10), when the condensate density vanishes at two points of the extended plane. To have an example convenient for analytical and numerical study, we introduce the complex variable $z = x + iy$ and consider the following special case of density profiles with two "holes":

$$\rho(\mathbf{x}) = 4\Big(\Big|\frac{Az}{1-Bz}\Big|^{\alpha} + \Big|\frac{1-Bz}{Az}\Big|^{\alpha}\Big)^{-2}, \tag{11}$$

with the parameters $\alpha > 0$, $A > 0$, $B \geq 0$. We introduce the curvilinear conformal coordinates $\mathbf{u} = (u, v)$ in the plane $\mathbf{x}$ by the analytic function

$$u + iv \equiv w = \ln\left(Az/[1 - Bz]\right). \tag{12}$$

The inverse mapping is specified by the expression $z = 1/[Ae^{-w} + B]$, and the coordinates (u, v) are located on a cylinder: $u \in (-\infty : +\infty)$, $v \in [0 : 2\pi)$. The Jacobian of this mapping is

$$J(\mathbf{u}) = |Ae^{-w}/(Ae^{-w} + B)^2|^2. \tag{13}$$

The density in terms of the new coordinates is given by the formula

$$\rho(\mathbf{u}) = \rho(u) = 1/\cosh^2(\alpha u). \tag{14}$$

The dimensionless equations of motion of vortices in terms of $\mathbf{u}_n$ have the form

$$\sigma_n J(\mathbf{u}_n)\rho(u_n)\dot{u}_n = \partial\tilde{H}_\alpha/\partial v_n, \qquad -\sigma_n J(\mathbf{u}_n)\rho(u_n)\dot{v}_n = \partial\tilde{H}_\alpha/\partial u_n. \tag{15}$$

The equation for the Green's function in the conformal variables preserves its simple structure:

$$-\nabla_{\mathbf{u}} \cdot \frac{1}{\rho(u)} \nabla_{\mathbf{u}} G(\mathbf{u}, \mathbf{u}_0) = 2\pi \delta(\mathbf{u} - \mathbf{u}_0). \tag{16}$$

It is supplemented by the 2π-periodic boundary conditions with respect to the coordinate v, zero asymptotics at $u \to +\infty$, and the condition of zero velocity circulation around the point $z = 0$, which implies

$$\lim_{u \to -\infty} \frac{\partial_u G(\mathbf{u}, \mathbf{u}_0)}{\rho(u)} = 0. \tag{17}$$

The solution is found by substituting

$$G(\mathbf{u}, \mathbf{u}_0) = \frac{g_\alpha(\mathbf{u}, \mathbf{u}_0)}{\cosh(\alpha u) \cosh(\alpha u_0)}. \tag{18}$$

The function $g_\alpha(\mathbf{u}, \mathbf{u}_0)$ satisfies the equation with constant coefficients,

$$[-\nabla_{\mathbf{u}}^2 + \alpha^2] g_\alpha(\mathbf{u}, \mathbf{u}_0) = 2\pi \delta(\mathbf{u} - \mathbf{u}_0). \tag{19}$$

The required solution is

$$g_\alpha(\mathbf{u}, \mathbf{u}_0) = \mathsf{g}(u - u_0, v - v_0) + \frac{1}{2\alpha} e^{-\alpha(u + u_0)}, \tag{20}$$

where the function $\mathsf{g}(U, V)$ even in both of its variables is given by the rapidly converging sum

$$\mathsf{g}(U, V) = \sum_{l=-\infty}^{+\infty} K_0(\alpha \sqrt{U^2 + (V + 2\pi l)^2}). \tag{21}$$

3. HAMILTONIAN

Composing the Hamiltonian of vortices with the use of the found Green's function, one can take into account the presence of an arbitrary integer number Q_0 of circulation quanta around the point $z=0$ by the respective modification of the asymptotic condition for the stream function at $u\to-\infty$. It should also be mentioned that at $B\neq 0$ several ($|Q_\infty|$) vortices can be placed near the point $w_\infty=\ln(A/B)+i\pi$, which corresponds to $z=\infty$. Since the Jacobian J diverges at this point, equations of motion (15) imply that these vortices will remain immobile at the point (u_∞,π). This way, arbitrary (integer) numbers Q_0 and Q_1 of circulation quanta around each of two finite zero-density points $z_0=0$ and $z_1=1/B$ can be provided, because the integers Q_0, Q_1 and Q_∞ satisfy the relation $Q_0+Q_1+Q_\infty+\sum_{n=1}^{N}\sigma_n=0$, where N is the number of mobile vortices. As a result, the dimensionless Hamiltonian is given by the expression

$$\tilde{H}_\alpha^{\{N\}}=\frac{1}{2}\sum_n\frac{\Lambda_0+\ln[\sqrt{(J(\mathbf{u}_n)/J_0)}/\cosh(\alpha u_n)]}{\cosh^2(\alpha u_n)}+\frac{1}{\alpha}\Big\{Q_0+\frac{Q_\infty}{(1+e^{2\alpha u_\infty})}+\sum_n\frac{\sigma_n}{(1+e^{2\alpha u_n})}\Big\}^2$$
$$+\sum_{n,m}{}'\frac{\sigma_n\sigma_m}{2}\frac{\mathsf{g}(u_n-u_m,v_n-v_m)}{\cosh(\alpha u_n)\cosh(\alpha u_m)}+Q_\infty\sum_n\sigma_n\frac{\mathsf{g}(u_n-u_\infty,v_n-\pi)}{\cosh(\alpha u_n)\cosh(\alpha u_\infty)},\qquad(22)$$

where $\Lambda_0=\mathsf{g}(\xi_*/\sqrt{J(0)},0)\gg 1$ is a large logarithm. The applicability of this formula requires that all numerators in the first sum not be small compared to unity. In the opposite case, the approximation of a point vortex fails.

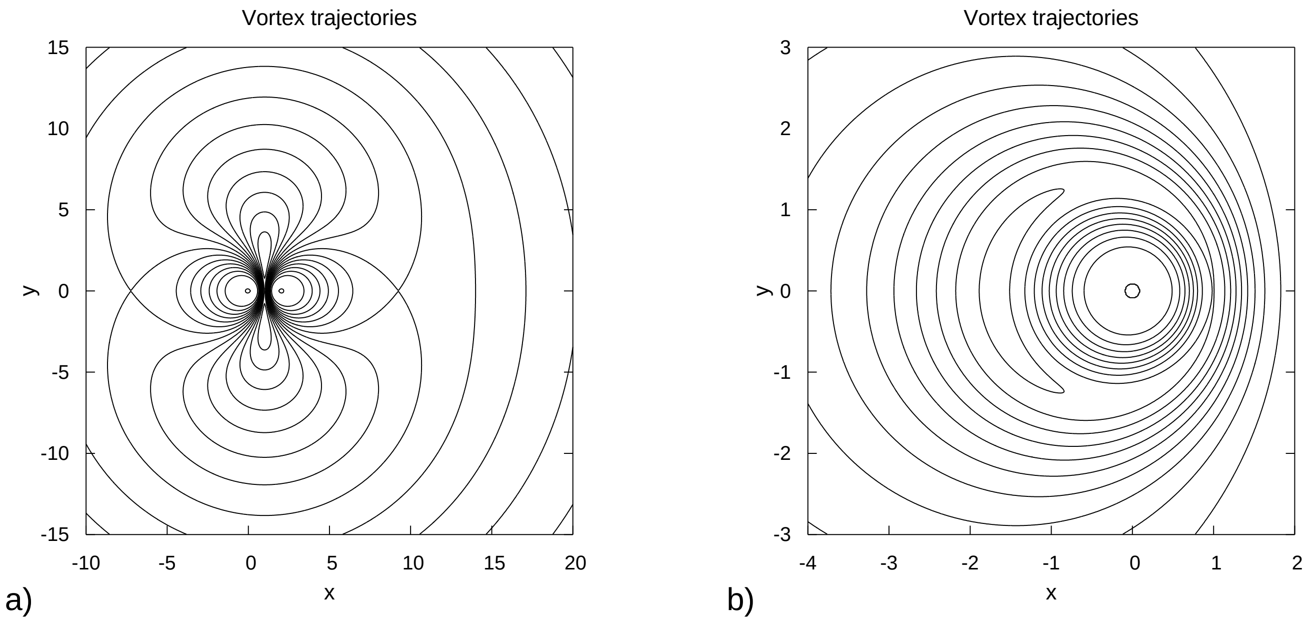

Figure 1: Examples of the trajectories of a single vortex in the plane for two sets of parameters: (a) $A = B = 0.5$, $Q_0 = 1$, $Q_1 = 1$, and (b) $A = 0.8$, $B = 0.2$, $Q_0 = 1$, $Q_1 = 0$.

In all numerical examples presented below, we set $\Lambda_0 = 7.0$, $\alpha = 2.0$, and all vortices have a positive sign, i.e., $\sigma_n = +1$.

To see a qualitative difference of this system from systems of type (9), we first briefly consider the dynamics of a single vortex. The trajectories of the vortex in the plane are the level contours of the Hamiltonian (22) at $N = 1$. Two examples of phase portraits with different sets of parameters A, B, Q_0, and Q_1 are shown in Fig.1.

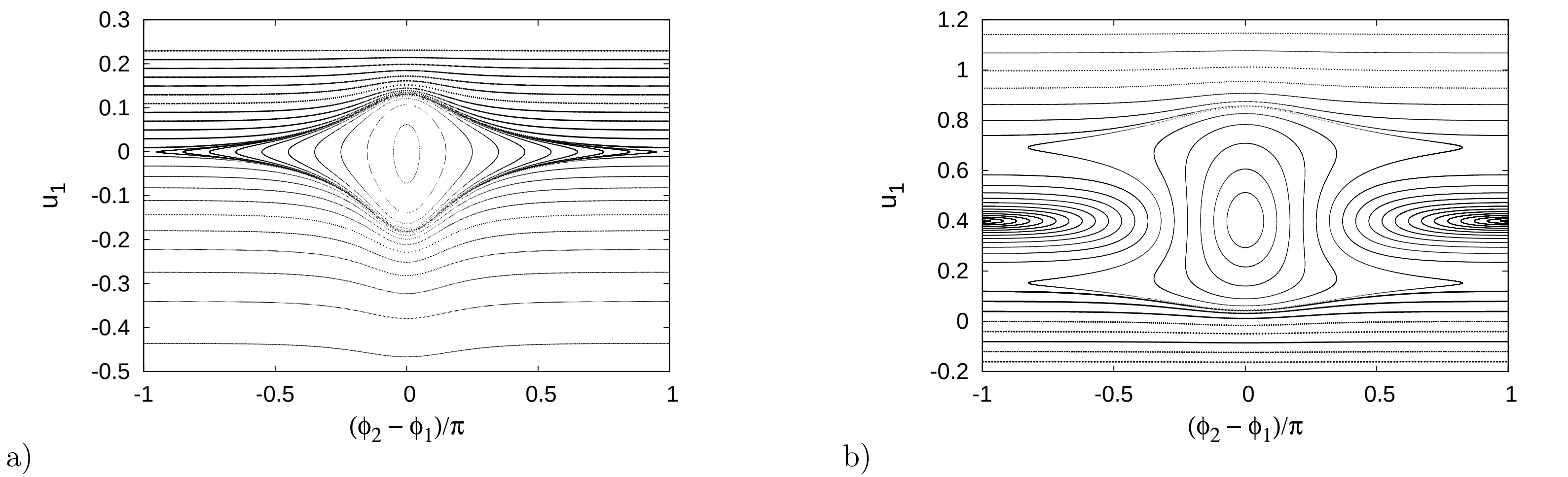

Figure 2: Phase trajectories of two vortices at $Q_0 = 0$ and the angular momentum (a) $M = 2\mu(0)$, (b) $M = 2\mu(0.4)$.

4. AXISYMMETRIC CASE

If $B = 0$, the system becomes axisymmetric: the Jacobian $J(u)$ is independent of the angular coordinate $v = \phi$, and the vortex Hamiltonian (22) is invariant with respect to a simultaneous shift of all variables v_n. Therefore, one more integral of motion emerges, the angular momentum

$$M = \sum_n \sigma_n \mu(u_n), \qquad \mu(u) = \int_{-\infty}^{u} J(s)\rho(s)ds. \tag{23}$$

The presence of the second conservation law makes the problem of motion of two vortices integrable. The projections of the phase trajectories of the system of two vortices on the $(\phi_2 - \phi_1, u_1)$ plane at a constant M value are exemplified in Fig.2 (without loss of generality, we put $A = 1$ as $B = 0$).

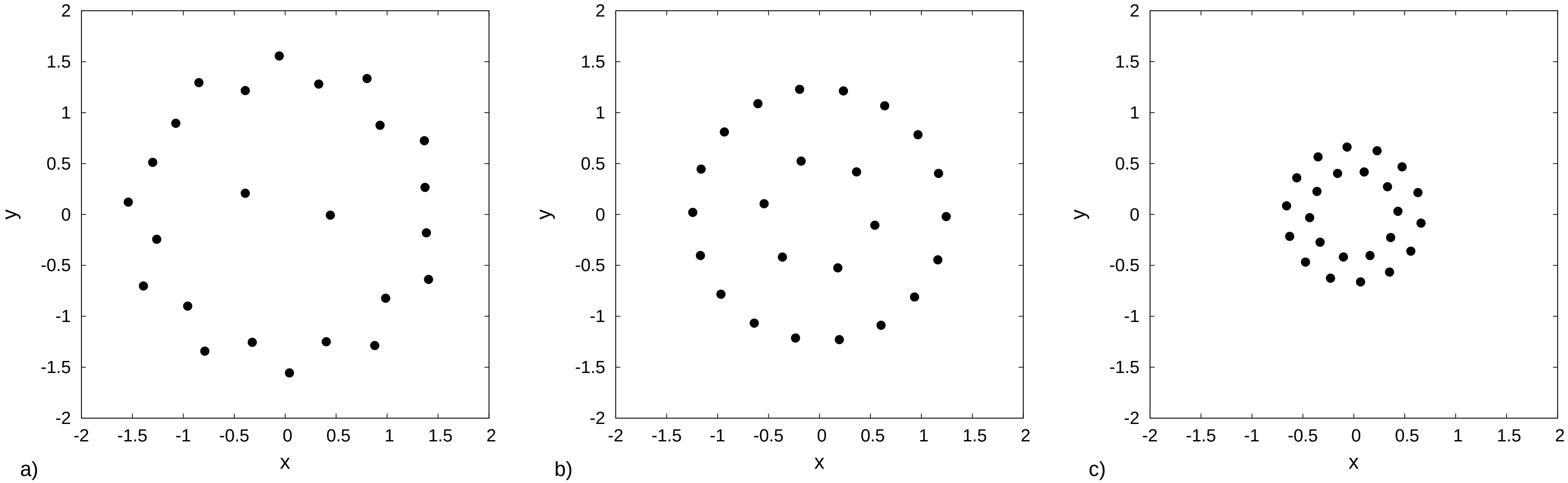

Figure 3: Examples of steadily rotating configurations of $N = 24$ vortices at $Q_0 = 12$ and the angular velocities a) $\Omega = 15$, b) $\Omega = 20$, c) $\Omega = 50$.

In addition, "rigid" rotating configurations of N vortices corresponding to the local minima of the function $\tilde{H}_{\alpha}^{\{N\}} + \Omega M$, where Ω is the angular velocity of rotation, become possible at the axial symmetry. The examples are shown in Fig.3. As is seen, the vortices are not always situated near concentric circles at equilibrium. It should also be mentioned that, with an increase in the rotation rate, the vortices leave the maximum-density region and accumulate inside the ring, which agrees qualitatively with the formation of a giant vortex.

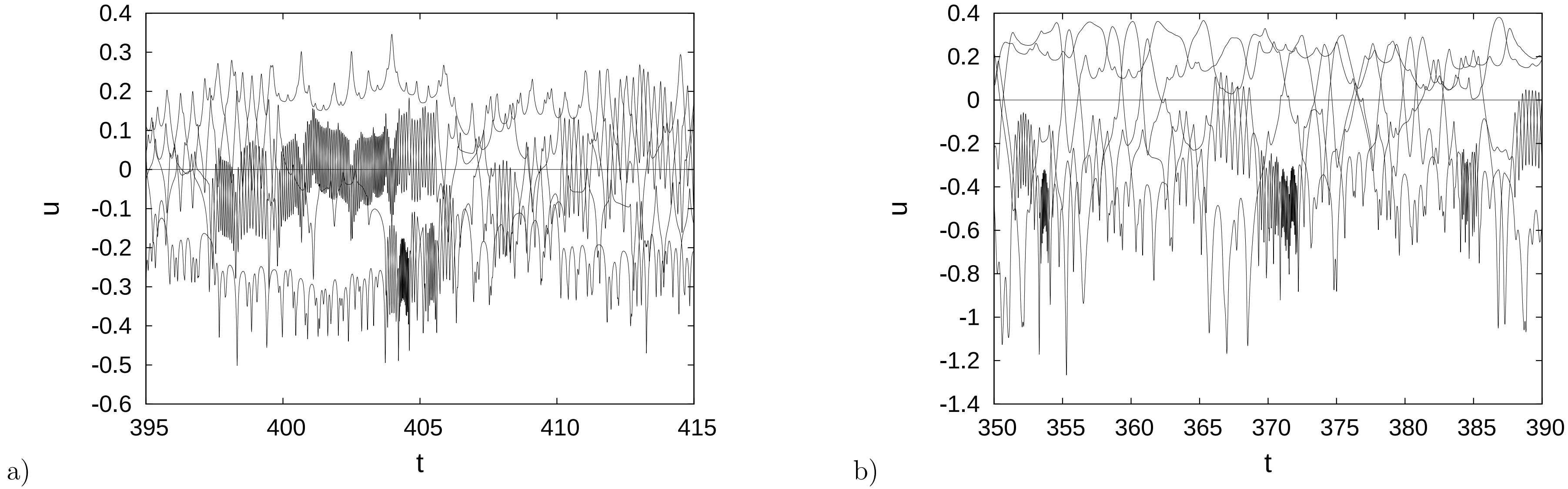

Figure 4: Chaotic dynamics of five vortices at: a) $Q_0 = 0$, b) $Q_0 = 12$.

The dynamics at $N \geq 3$ in a wide range of initial values exhibits the features of chaos. Chaos is caused by the formation of pretty tight and thus rapidly rotating vortex pairs (of the same sign). The very fact of the formation of a pair, its lifetime, size, rotation phase, and other characteristics appear to be almost unpredictable owing to the interaction with other vortices. Two numerical examples of the behavior of five vortices are presented in Fig.4.

5. VORTICES ON A CURVED SURFACE

One can imagine the situation where a 3D trap potential forms a narrow curved "canyon", so that the Bose-Einstein condensate at equilibrium is not (inhomogeneously) planar, as was assumed so far, but forms a relatively narrow shell near some curved surface in the 3D space. An interesting generalization of the above theory is the description of the dynamics of quantum vortices in such systems if the variables u and v are regarded as the conformal coordinates on the curved surface $\mathbf{r} = \mathbf{S}(u, v)$. It is easily verified that the equations of motion (15) for u_n and v_n preserve its structure and so does the Hamiltonian (22) (with a possible restriction of $Q_\infty = 0$, depending on the type of the surface). An essential difference is that the function $J(\mathbf{u})$ is now not the squared absolute value of an analytic function $z'(w)$ but a quite arbitrary conformal factor. In other words, the metric of the surface is

$$(d\mathbf{S})^2 = J(\mathbf{u})(du^2 + dv^2). \tag{24}$$

We consider as an example a unit sphere whose stereographic projection on the plane is determined by the formula

$$z + c = \tan(\Theta/2)e^{i\Phi}, \tag{25}$$

where Θ is the polar angle on the sphere, Φ is the azimuthal angle, and c is an arbitrary complex constant. The conformal factor in terms of the variables $\mathbf{x}$ is

$$\mathsf{J}(\mathbf{x}) = 4/(1 + |x + iy + c|^2)^2. \tag{26}$$

It is worth mentioning that exactly this factor must be added to the left-hand sides of Eqs. (1) if one chooses the variables x and y for the description of the vortex dynamics on a sphere. Accordingly, the logarithmically diverging terms in the Hamiltonian must be regularized with the use of $\mathsf{J}(\mathbf{x})$.

The conformal factor in terms of $\mathbf{u}$ is the product $\mathsf{J}(\mathbf{x}(\mathbf{u}))\cdot|z'(w)|^2$, which yields in our case

$$J(\mathbf{u}) = \frac{4A^2e^{-2u}}{\{|Ae^{-w}+B|^2+|c(Ae^{-w}+B)+1|^2\}^2}. \tag{27}$$

Since this expression is finite everywhere, all vortices on the sphere at finite u values are mobile, and hence $Q_\infty = 0$ should be put in Hamiltonian (22).

Formula (27) is greatly simplified at $B=0$ and $c=0$, i.e., in the presence of the axial symmetry: $J(\mathbf{u}) = 1/\cosh^2(u-u_{\rm eq})$, with $u=u_{\rm eq}$ corresponding to the equator. As in the planar case, the axial symmetry implies the conservation of momentum.

It is interesting that, even in the case of $u_{\rm eq}=0$, where the 2D density profile is symmetric with respect to the equatorial plane, the Hamiltonian of an odd number of vortices still is not symmetric with respect to the inversion $u_n \to -u_n$, since the velocity circulation quanta around the poles cannot be equal to each other owing to the condition $Q_{\rm North}+Q_{\rm South}+\sum_n \sigma_n = 0$.

6. SUMMARY

To summarize, a new, relatively simple, mathematically convenient, and rather rich model has been proposed which allows advancing in the understanding of the mechanics of vortices in spatially inhomogeneous 2D systems. A number of numerical examples have been presented. Many other situations approximately corresponding to various actual experiments can be studied by varying the parameters of the model. In particular, nonlinear oscillations of many vortices near steady configurations and the dynamics of oppositely oriented vortices remained beyond the scope of this work.

Seemingly, many qualitative properties of the model Green's function found in this work persist for a wider class of the dependences $\rho(u)$, when $\alpha^2 \neq$ const in Eq.(19). This question requires a separate investigation.

THANK YOU FOR YOUR ATTENTION!

BEST WISHES TO E.A.!