Solitons, Collapses and Turbulence

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SCT - 2017

Landau Institute for Theoretical Physics (Chernogolovka, Russia) , May 2017

Как этот мир велик в лучах рабочей лампы,
А в памяти очах как бесконечно мал…
(Ш.Бодлер, пер. М.Цветаевой)

100 years of General Relativity

Collision of gravitational and electromagnetic solitons with strong electromagnetic waves of arbitrary amplitudes and profiles in the expanding background space-time

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- ***Gravitational and electromagnetic waves in (linear approximation)***

  -- "Gertsenstein effect" (M.E. Gertsentein 1961)
  -- mutual transformation of waves (GA & Sibgatullin 1974)

- ***Nonlinear waves in the expanding universe (exact solutions):***

  -- the symmetric Kasner space-time as a background for waves
  -- travelling electromagnetic waves of arbitrary amplitudes
  -- gravitational and electromagnetic solitons
  -- collision of solitons with travelling waves

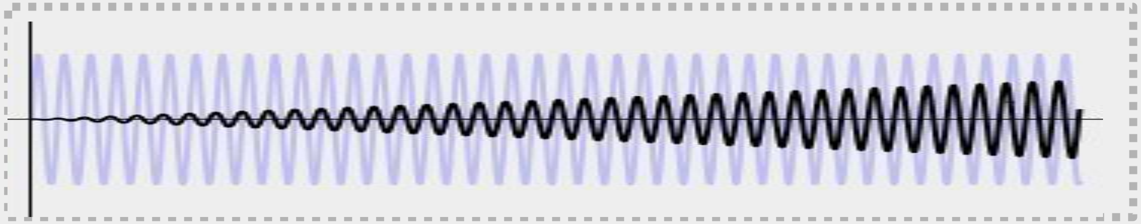
Einstein - Maxwell equations

$$\begin{cases} R_{ik} = -\frac{\varkappa}{4\pi}\left(F_{il}F_k{}^l - \frac{1}{4}g_{ik}F_{lm}F^{lm}\right) \\ \nabla_k F^{ik} = 0, \qquad \nabla_{[i}F_{jk]} = 0 \end{cases}$$

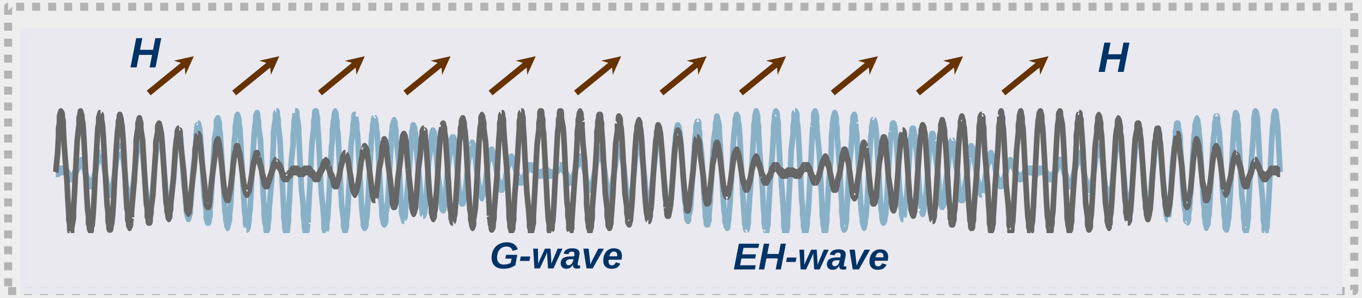
Gravitational and electromagnetic waves in external fields

(small amplitudes -- linear approximation):

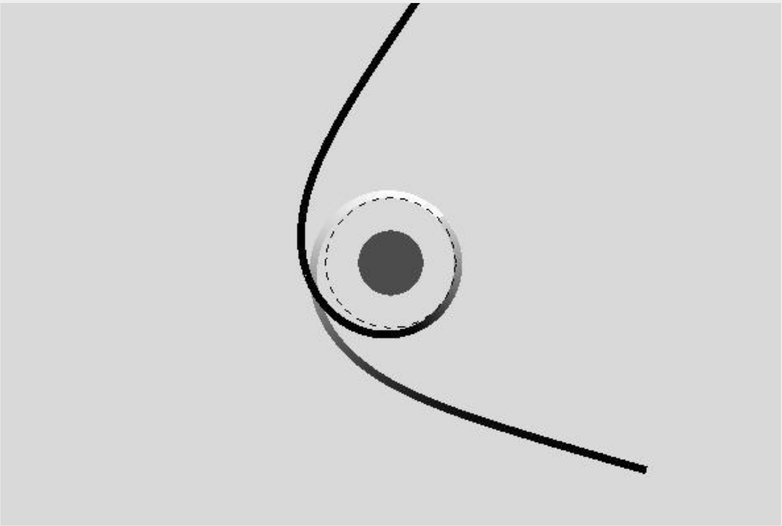
Gertsenstein M.E. (1961):



Sibgatullin N.R. & GA (1974):



GA (1976):



Einstein - Maxwell equations

$$\begin{cases} R_{ik} = -\frac{\varkappa}{4\pi}\Big(F_{il}F_k{}^l - \frac{1}{4}g_{ik}F_{lm}F^{lm}\Big) \\ \nabla_k F^{ik} = 0, \qquad \nabla_{[i}F_{jk]} = 0 \end{cases}$$

Complex Ernst potentials

$$\begin{matrix} g_{ik}(x^1, x^2) \\ F_{ik}(x^1, x^2) \end{matrix} \quad \Longleftrightarrow \quad \begin{matrix} \mathcal{E}(x^1, x^2), \\ \Phi(x^1, x^2) \end{matrix}$$

Wave fields

$$u = t - x, \qquad v = t + x$$

Complex Ernst equations:

$$\begin{cases} (\mathrm{Re}\,\mathcal{E} + \Phi\overline{\Phi})\left(2\mathcal{E}_{uv} + \frac{\alpha_u}{\alpha}\mathcal{E}_v + \frac{\alpha_v}{\alpha}\mathcal{E}_u\right) - \left(\mathcal{E}_u + 2\overline{\Phi}\Phi_u\right)\mathcal{E}_v - \left(\mathcal{E}_v + 2\overline{\Phi}\Phi_v\right)\mathcal{E}_u = 0 \\ (\mathrm{Re}\,\mathcal{E} + \Phi\overline{\Phi})\left(2\Phi_{uv} + \frac{\alpha_u}{\alpha}\Phi_v + \frac{\alpha_v}{\alpha}\Phi_u\right) - \left(\mathcal{E}_u + 2\overline{\Phi}\Phi_u\right)\Phi_v - \left(\mathcal{E}_v + 2\overline{\Phi}\Phi_v\right)\Phi_u = 0 \\ \alpha_{uv} = 0 \end{cases}$$

Inverse scattering approach to Einstein - Maxwell fields [1)]

Dynamical equations $\mathbf{U}(\xi,\eta),\ \mathbf{V}(\xi,\eta)$

$$\begin{cases} \partial_\eta \mathbf{U} + \partial_\xi \mathbf{V} + \dfrac{[\mathbf{U},\mathbf{V}]}{\xi-\eta} = 0 \\ \partial_\eta \mathbf{U} - \partial_\xi \mathbf{V} = 0 \end{cases} \qquad \Bigg\| \qquad \begin{array}{ll} \mathbf{U}\cdot\mathbf{U} = i\,\mathbf{U}, & \mathrm{Tr}\,\mathbf{U} = i \\ \mathbf{V}\cdot\mathbf{V} = i\,\mathbf{V}, & \mathrm{Tr}\,\mathbf{V} = i \end{array}$$

Associated spectral problem $\mathbf{\Psi} = \mathbf{\Psi}(\xi,\eta,w)$

$$\begin{cases} 2i(w-\xi)\partial_\xi \mathbf{\Psi} = \mathbf{U}(\xi,\eta)\,\mathbf{\Psi} \\ 2i(w-\eta)\partial_\eta \mathbf{\Psi} = \mathbf{V}(\xi,\eta)\,\mathbf{\Psi} \end{cases} \qquad \Bigg\| \qquad \begin{array}{ll} \mathbf{U}\cdot\mathbf{U} = i\,\mathbf{U} & \mathrm{tr}\,\mathbf{U} = i \\ \mathbf{V}\cdot\mathbf{V} = i\,\mathbf{V} & \mathrm{tr}\,\mathbf{V} = i \end{array}$$

Matrix integral $\mathbf{K}(w)$

$$\begin{cases} \mathbf{\Psi}^\dagger\,\mathbf{W}\,\mathbf{\Psi} = \mathbf{K}(w) \\ \mathbf{K}^\dagger(w) = \mathbf{K}(w) \end{cases} \qquad \Bigg\| \qquad \frac{\partial \mathbf{W}}{\partial w} = \mathbf{\Omega}, \qquad \mathbf{\Omega} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

"Dressing" method for constructing of solitons: $(\overset{o}{g}_{ab}, \overset{o}{A}_b, \overset{o}{\mathbf{\Psi}}) \;\rightarrow\; (g_{ab}, A_b \mathbf{\Psi})$

$$\mathbf{\Psi} = \chi\cdot\overset{o}{\mathbf{\Psi}}, \qquad \chi = \mathbf{I} + \sum_{k=1}^{N}\frac{\mathbf{R}_k(\xi,\eta)}{w-w_k}, \qquad \chi^{-1} = \mathbf{I} + \sum_{k=1}^{N}\frac{\mathbf{S}_k(\xi,\eta)}{w-\widetilde{w}_k}$$

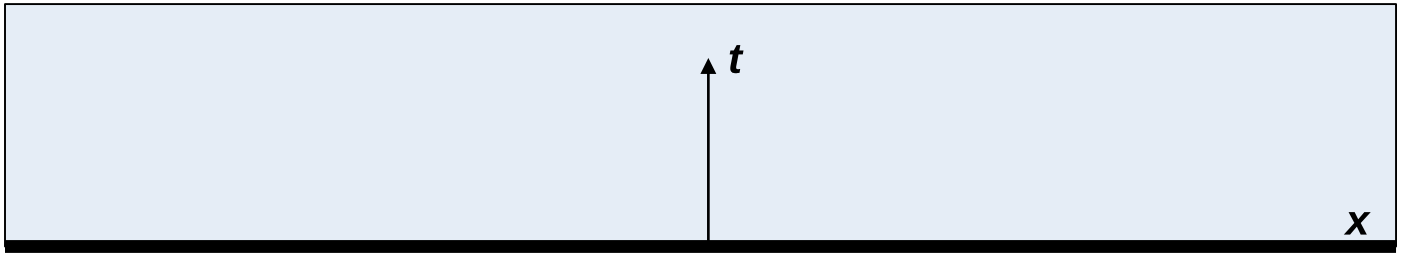
1) *GA, JETP Lett 1980*

Background for waves: the symmetric Kazner vacuum universe

$$ds^2 = -\frac{1}{\sqrt{t}}(dt^2 - dx^2) + t(dy^2 + dz^2)$$

$$\xi = -x + t = u$$
$$\eta = -x - t = -v$$

Space-time:



Solution of the spectral problem:

$$\Psi = \frac{1}{2}\left(\frac{w-\xi_0}{w-\xi}\right)^{1/2}\begin{pmatrix}1\\ it_0\end{pmatrix}\otimes\left(1, -\frac{i}{t_0}\right) + \frac{1}{2}\left(\frac{w-\eta_0}{w-\eta}\right)^{1/2}\begin{pmatrix}1\\ -it_0\end{pmatrix}\otimes\left(1, \frac{i}{t_0}\right)$$

$$\Psi^{-1} = \frac{1}{2}\left(\frac{w-\xi}{w-\xi_0}\right)^{1/2}\begin{pmatrix}1\\ it_0\end{pmatrix}\otimes\left(1, -\frac{i}{t_0}\right) + \frac{1}{2}\left(\frac{w-\eta}{w-\eta_0}\right)^{1/2}\begin{pmatrix}1\\ -it_0\end{pmatrix}\otimes\left(1, \frac{i}{t_0}\right)$$

Gravitational and electromagnetic soliton waves on the symmetric Kasner background

$$\begin{cases} ds^2 = -f\,du\,dv + t\Big[h_{yy}dy^2 + 2h_{yz}dydz + h_{zz}dz^2\Big], \\ A_i = \{0,\,0,\,A_y,\,A_z\}, \end{cases} \qquad \Bigg\| \quad \begin{matrix} \ell + is, \\ c, \\ d \end{matrix}$$

$$h_{yy} = \frac{(\mathcal{G} - 4isc)(\overline{\mathcal{G}} + 4is\bar{c})}{\mathcal{D}\overline{\mathcal{D}}}, \qquad h_{yz} = -\frac{4s\,(c\,\overline{\mathcal{G}} + \bar{c}\,\mathcal{G})}{\mathcal{D}\overline{\mathcal{D}}}, \qquad \{A_y,\,A_z\} = \Re\{\Phi, \widetilde{\Phi}\},$$

$$\Phi = -2sd\,\mathcal{F}/\mathcal{D},$$

$$h_{zz} = \frac{(\mathcal{G} + 4isc)(\overline{\mathcal{G}} - 4is\bar{c})}{\mathcal{D}\overline{\mathcal{D}}}, \qquad f = \frac{1}{\sqrt{t}} \cdot \frac{\mathcal{D}\overline{\mathcal{D}}}{4\,r_+ r_-} \qquad \widetilde{\Phi} = 2isd\,\widetilde{\mathcal{F}}/\mathcal{D}$$

$$\mathcal{D} = \overline{K}_+ K_- - c\,\bar{c}\,K_+ \overline{K}_- + d\,\bar{d}\,\overline{K}_+ \overline{K}_-,$$

$$\mathcal{G} = K_+ K_- - c\,\bar{c}\,\overline{K}_+ \overline{K}_- + d\,\bar{d}\,K_+ \overline{K}_-,$$

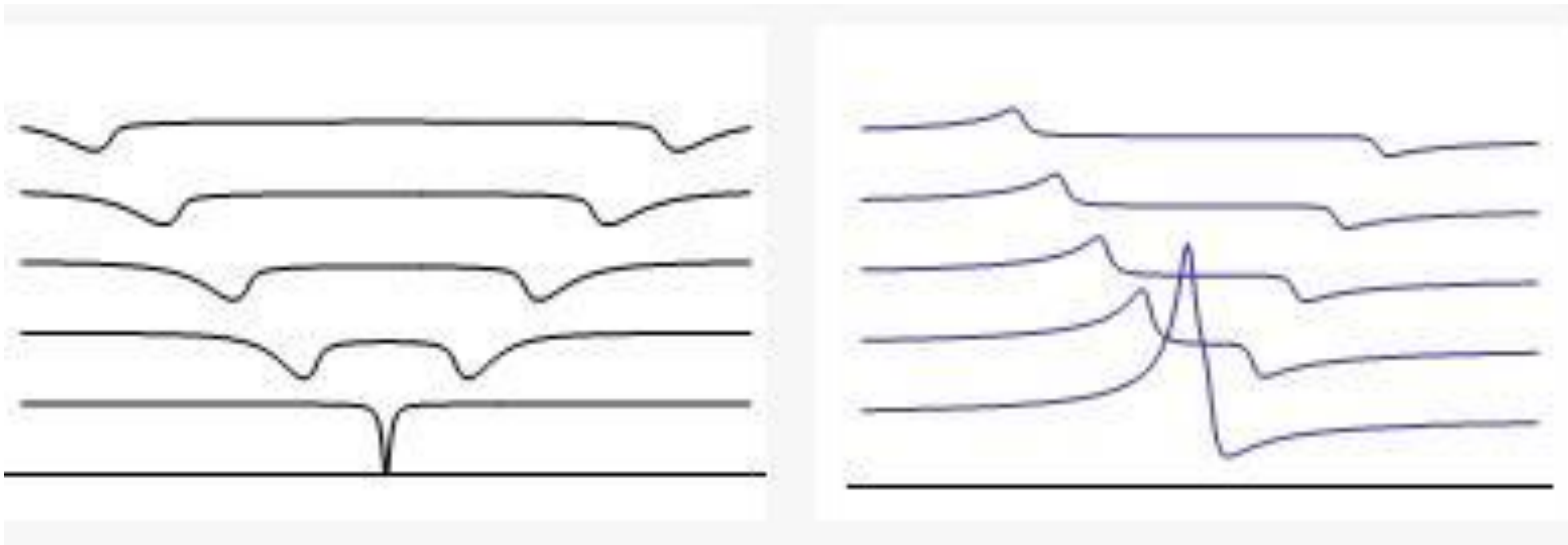
$$\mathcal{F} = \overline{K}_+ - \bar{c}\overline{K}_-, \qquad \widetilde{\mathcal{F}} = \overline{K}_+ + \bar{c}\overline{K}_-.$$

$$K_+ = \sqrt{R_+} + \frac{is}{\sqrt{R_+}}, \qquad K_- = \sqrt{R_-} + \frac{is}{\sqrt{R_-}}$$

$$R_+ = \ell - u + r_+, \qquad r_+ = \sqrt{(\ell - u)^2 + s^2},$$

$$R_- = \ell + v + r_-, \qquad r_- = \sqrt{(\ell + v)^2 + s^2}.$$

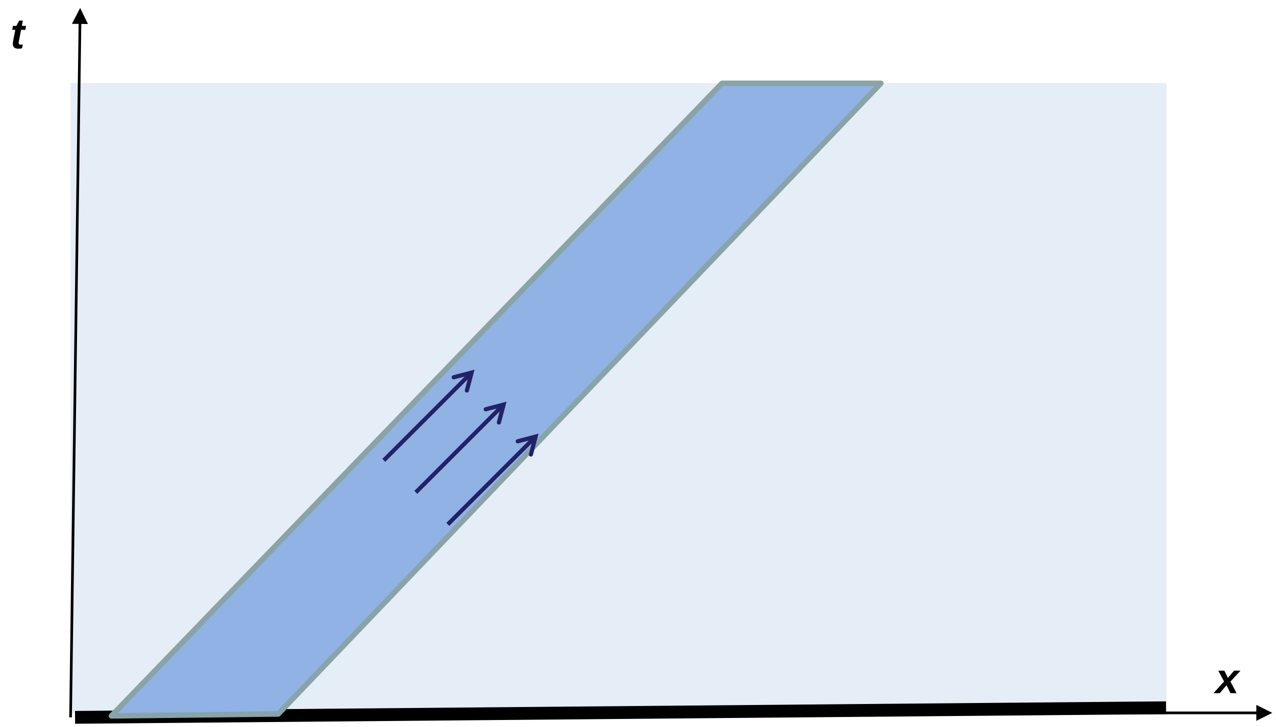
One-soliton solution on the symmetric Kazner background



$\mathrm{Im}\, c = 0, \quad d = 0$ $\qquad$ $\mathrm{Re}\, c = 0, \quad d = 0$

Plane electromagnetic waves on the symmetric Kazner background [1)]

$$\begin{cases} ds^2 = -\dfrac{1}{\sqrt{t}} e^{4\mathcal{B}(u)} du\, dv + t(dy^2 + dz^2), \\ A_i = \{0,\, 0,\, \phi(u),\, 0\}, \quad \mathcal{B}(u) = \int \phi'^2(u) du \end{cases}$$



1) *GA, Classical and Quantum Gravity 2015*

Solutions of the linear system for a travelling electromagnetic wave

$$\Psi_L = \begin{pmatrix} \frac{1}{\sqrt{w-\eta}} & \frac{e^{S_+(\xi,w)}}{\sqrt{w-\xi}} & \frac{e^{-S_+(\xi,w)}}{\sqrt{w-\xi}} \\ -\frac{i}{\sqrt{w-\eta}} & \frac{ie^{S_+(\xi,w)}}{\sqrt{w-\xi}} & \frac{ie^{-S_+(\xi,w)}}{\sqrt{w-\xi}} \\ 0 & 2i\sqrt{2}e^{S_+(\xi,w)}\left(1-\frac{\phi_+(\xi)}{\sqrt{w-\xi}}\right) & -2i\sqrt{2}e^{-S_+(\xi,w)}\left(1+\frac{\phi_+(\xi)}{\sqrt{w-\xi}}\right) \end{pmatrix}$$

$$S_+(\xi,w) = \int_{\xi_0}^{\xi} \frac{\phi'_+(\chi)}{\sqrt{w-\chi}}\,d\chi$$

Nonlinear interaction of gravitational and electromagnetic solitons with a non-soliton electromagnetic wave of arbitrary profile [1)]

Solution parameters: $\ell + is,\ c,\ d \quad \text{and} \quad \phi(u)$

Behaviour of a soliton in the interaction region:

$$c \to \widehat{c}(u), \qquad \Phi \to \Phi + \phi(u),$$
$$d \to \widehat{d}(u), \qquad \widetilde{\Phi} \to \widetilde{\Phi} + i\phi(u),$$
$$f \to e^{4\mathcal{B}(u)} f$$

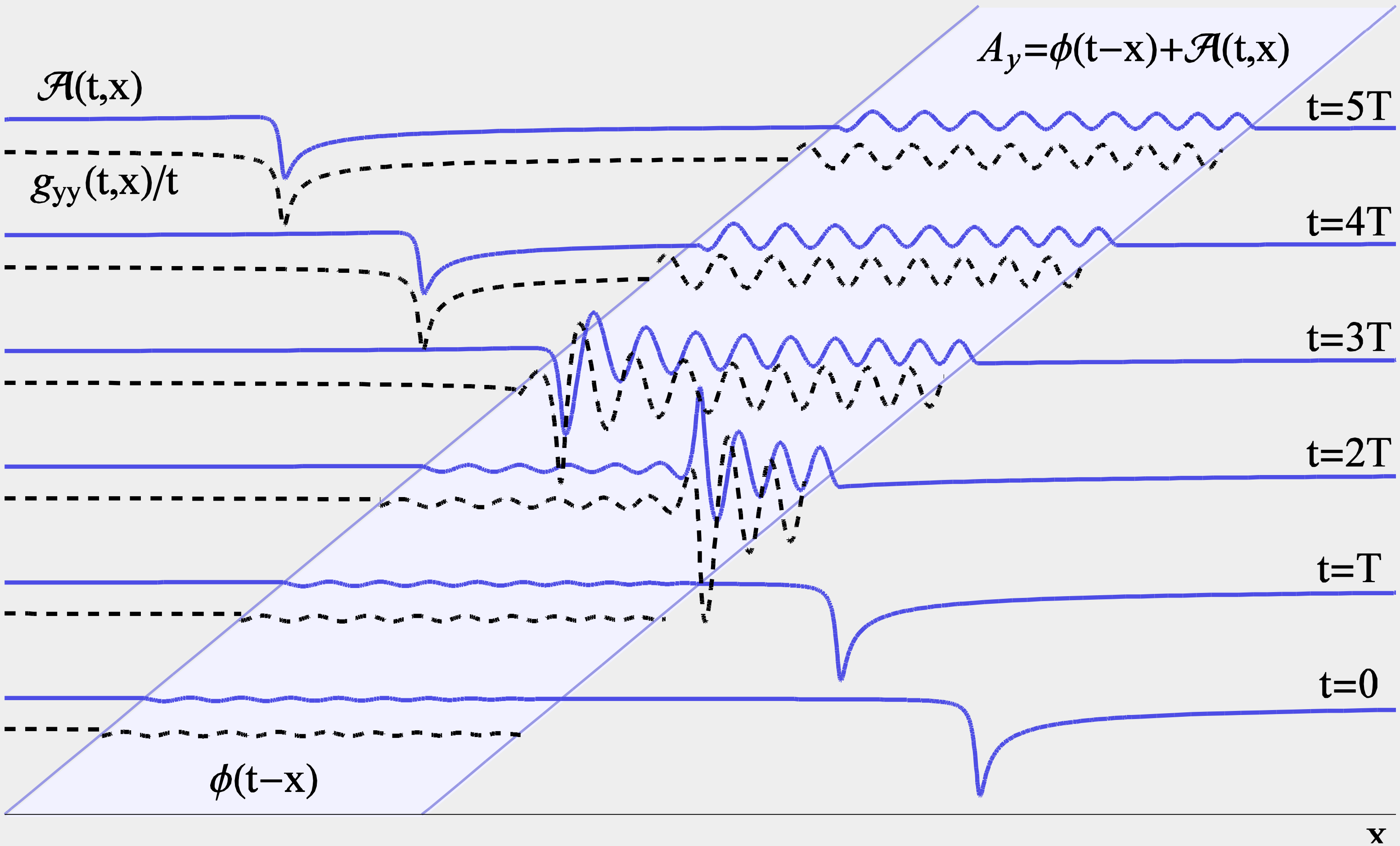
Evolution of the soliton parameters:

$$\widehat{c} = c \cosh S(u) - id \sinh S(u),$$
$$\widehat{d} = ic \sinh S(u) + d \cosh S(u),$$

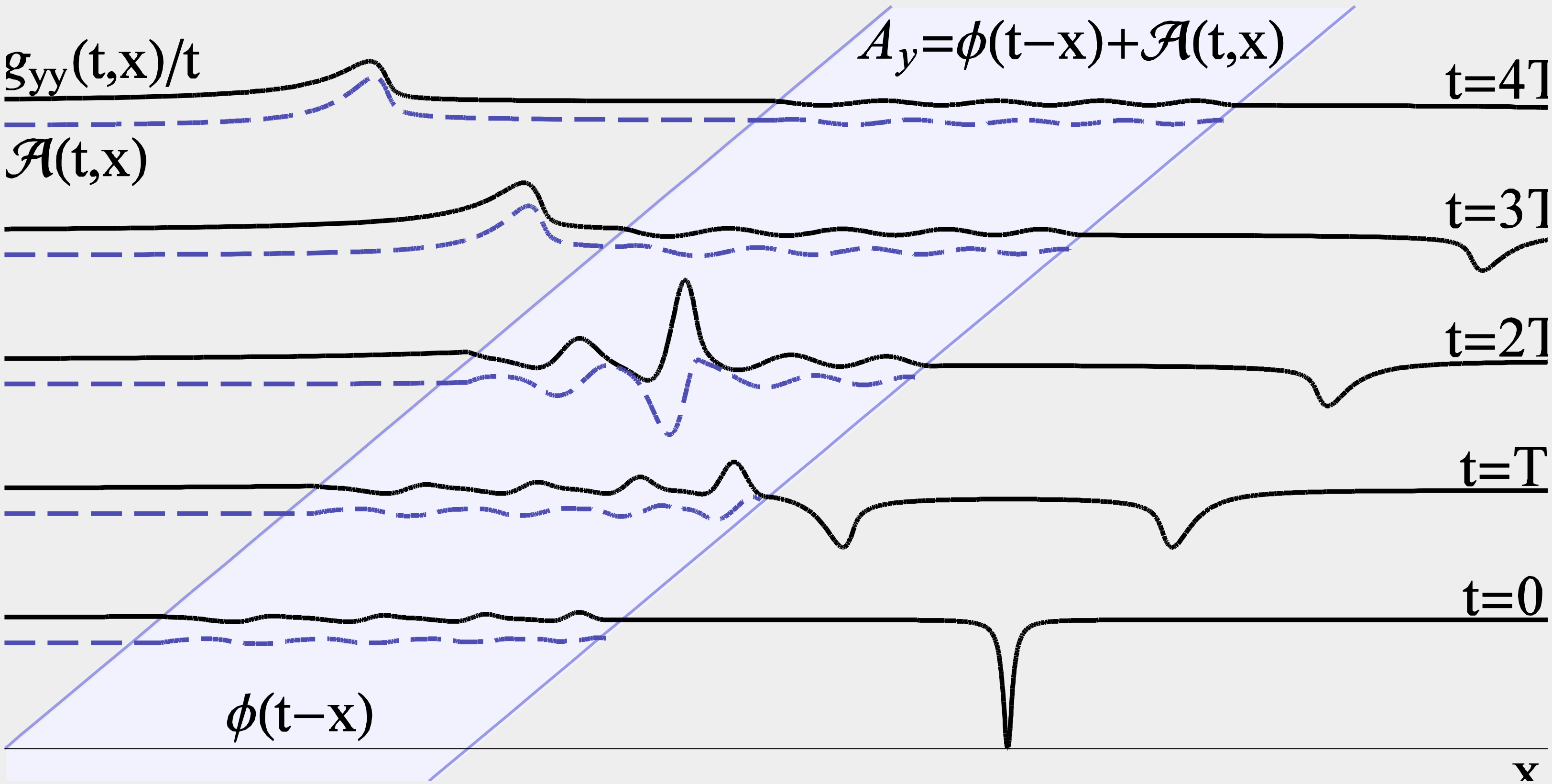
$$S(u) = \sqrt{2} \int\limits_{u_1}^{u} \frac{\phi'(u)du}{\sqrt{w_o - u}}$$

1) GA, Physical Review D (2016)

Collision of electromagnetic soliton with non-soliton electromagnetic wave



Collision of vacuum soliton with travelling electromagnetic wave



Thank you