Equilibrium configurations of an uncharged conducting liquid jet in a transverse electric field

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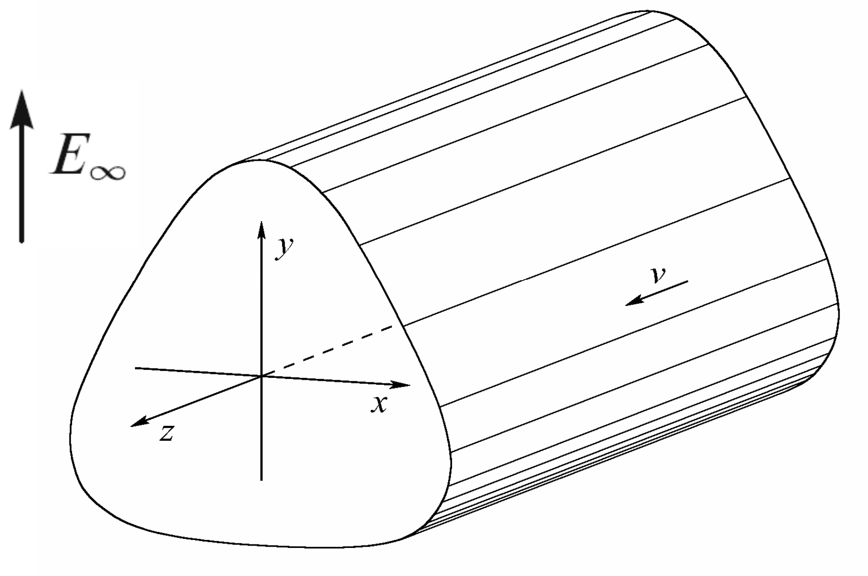
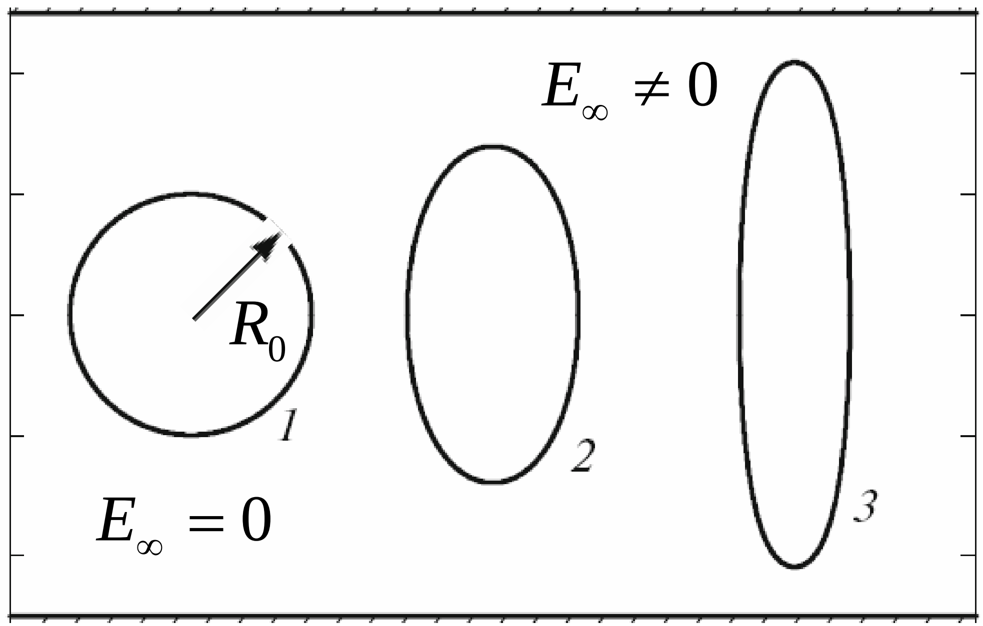
Let us place an uncharged cylindrical jet of a conducting liquid in a transverse electric field E_∞.

In the absence of the electric field, the jet has a circular cross section.

When $E_\infty \neq 0$, the jet is deformed under the action of electrostatic forces: its cross section is stretched along the lines of force of the field.

Our aim is to find a new equilibrium state of the system, where the electrostatic forces are counterbalanced by capillary forces at the deformed surface of the jet. Also, we will find the conditions under which equilibrium solutions do not exist and the jet splits into two.

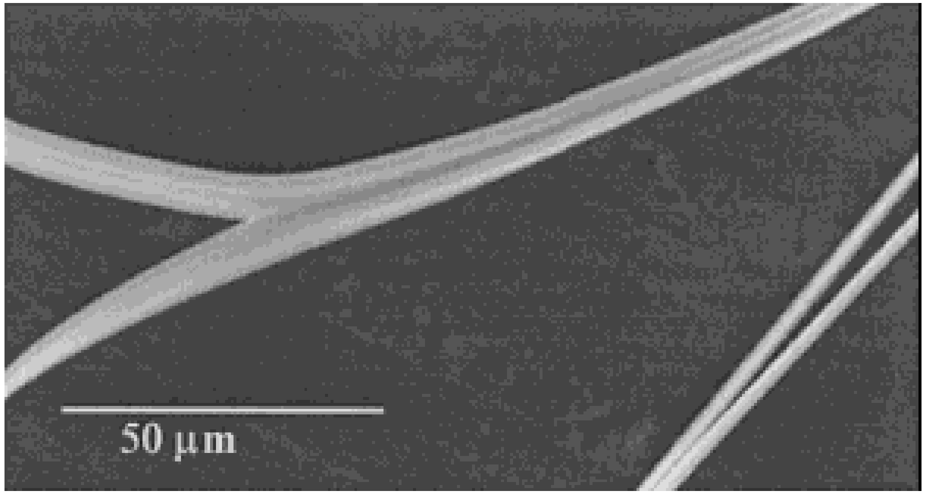




Cross sections of the jet.

From the applied point of view, interest in this electrostatic problem is associated with the possibility of controlled splitting of jets by an applied transverse electric field. This phenomenon is possible to use for the production of polymer microfibers [1]. Longitudinal splitting of jets under an electric field was observed in the experiments [2–5].

[3]:



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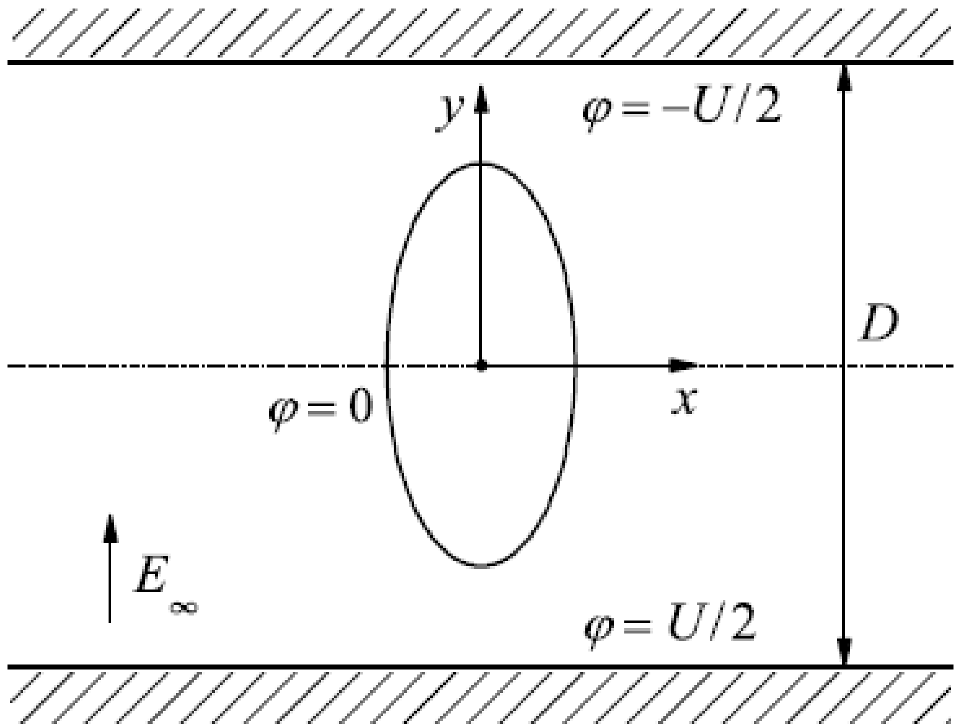
Geometry of the problem

An uncharged cylindrical jet of a conducting liquid is placed between two planar electrodes to which a potential difference U is applied. The electrodes are situated at a distance D from each other.

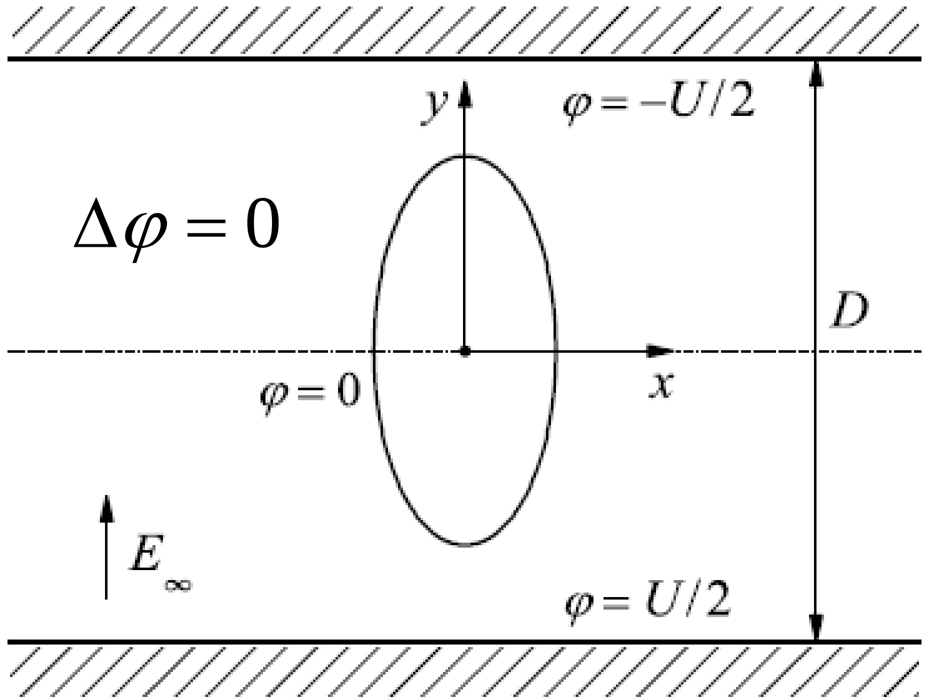
The liquid is at rest in the system of coordinates moving together with the jet.



The problem has plane symmetry – the surface of the jet is invariant with respect to the translation along its axis (as a result of the action of electrostatic forces, only the cross section of the jet is deformed). In this situation, all the quantities depend only on the pair of coordinates x and y that correspond to the plane of the jet cross section.

1. Original equations

The problem of finding possible equilibrium configurations of the jet is difficult to solve analytically. This is associated with the necessity to determine the electric field distribution around a jet whose shape is unknown and is determined by the essentially nonlinear condition of balance between electrostatic and capillary forces.



The electric field potential φ satisfies the 2D Laplace equation:

$$\varphi_{xx} + \varphi_{yy} = 0.$$

It should be solved together with the following conditions ($E_\infty = U/D$):

$\varphi = 0$ on the free surface,

$\varphi = \mp U/2$ on the electrodes $y = \pm D/2$,

$\varphi \to -E_\infty y$ at infinity $|x| \to \infty$.

2. Original equations

The equilibrium shape of the free surface is determined by the pressure balance condition:

$$\boxed{\varepsilon_0(\varphi_x^2 + \varphi_y^2)/2 + T\kappa + \Delta P = 0.}$$

The first term corresponds to the electrostatic pressure, and the second, to the capillary pressure (ε_0 is the dielectric constant, κ is the curvature, T is the surface tension coefficient, ΔP has the meaning of the pressure difference between the inside and outside of the liquid).

It should be noted that this electrostatic problem is analogous to the problem of the shape of a two-dimensional gas bubble moving in an ideal liquid for which a particular zero-parameter solution has been found earlier [E.B. McLeod, 1955].

Dimensionless notations

Let us pass to dimensionless variables by the substitutions

$$x \to x \cdot x_{\max}, \quad y \to y \cdot x_{\max}, \quad \varphi \to \varphi \cdot E_{\max} x_{\max},$$

where $E_{\max}$ is the maximum value of the electric field that is attained on the surface of the jet at the points $\{x, y\} = \{0, \pm y_{\max}\}$, and $x_{\max}$, $y_{\max}$ are the distances from the axis of the jet to its boundary in the directions x and y.

We introduce, for convenience, the following dimensionless combinations:

$$p_S = T/\Delta P x_{\max}, \qquad p_E = \varepsilon_0 E_{\max}^2 / 2\Delta P,$$

$$r_0 = R_0 / x_{\max}, \qquad d = D/x_{\max}, \qquad e = E_{\max}/E_\infty.$$

The pressure balance condition rewrites as

$$\boxed{p_E \left(\varphi_x^2 + \varphi_y^2\right) + p_S \kappa + 1 = 0.}$$

The conditions on the electrodes and at infinity take the form

$$\varphi = \mp d/2e, \qquad y = \pm d/2,$$

$$\varphi \to -y/e, \qquad |x| \to \infty.$$

1. Conformal variables

We introduce the complex potential $\Phi = \varphi - i\psi$ of the electric field, which is an analytic function of a complex variable $z = x + iy$. The function ψ is harmonically conjugated to the potential φ ; the condition $\psi = \mathrm{const}$ defines the electric field force lines. In addition, we introduce a complex strength of the field, $W = d\Phi/dz = \varphi_x - i\varphi_y$. Let us represent it as

$$W = -Ee^{-i\theta},$$

$$E = |W| = \sqrt{\varphi_x^2 + \varphi_y^2}, \qquad \theta = \arg W = \arctan\left(\varphi_y/\varphi_x\right).$$

It is convenient to take the complex electric field strength W as the unknown function and the complex potential Φ as an independent variable. The latter corresponds to the conformal mapping of the domain bounded by the electrodes and by the free surface of the jet into the strip

$$-d/2e \le \varphi \le d/2e, \qquad -\infty < \psi < \infty.$$

The surface of the jet is mapped onto the segment $|\psi| < 1, \quad \varphi = 0.$

2. Conformal variables

The analytic function W should be found. The boundary conditions for W take the form:

$$p_E E^2 - p_S E\theta_\psi + 1 = 0, \qquad \varphi = 0, \qquad |\psi| \le 1,$$

$$\theta = \pi/2, \qquad \varphi = 0, \qquad |\psi| > 1,$$

$$\theta = \pi/2, \qquad \varphi = -d/2e,$$

$$W \to i/e, \qquad |\psi| \to \infty.$$

Thus, the original problem with unknown boundary is reduced to a much simpler problem on a strip.

Construction of exact solutions

In the simplest case of a perfectly conducting circular cylinder placed in a uniform external field ($D \to \infty$), the following relation between the electric field strength and its inclination angle holds on its surface:

$$\boxed{E = \sin\theta.}$$

Suppose that this relation remains valid in the situation where the jet is deformed by the electrostatic pressure and D is arbitrary (see also the magnetic shaping problem [J.A. Shercliff, 1981]). In his situation the force balance condition takes the form of an ordinary differential equation

$$\boxed{p_E \sin^2\theta - p_S \theta_\psi \sin\theta + 1 = 0.}$$

Its solution has the form

$$\theta(\psi) = \pi/2 + \arcsin\left(B \tanh(B\psi / p_S) / p_E\right).$$

Then we get for the complex field strength on the free surface:

$$W = -\sin\theta(\psi) \cdot e^{-i\theta(\psi)}.$$

The construction of the analytic continuation of W from the free surface to the domain outside the liquid yields

$$W(\Phi) = i\left[1 - B\tan\left(B\Phi / p_S\right) \Big/ \sqrt{p_E^2 + B^2\tan^2\left(B\Phi / p_S\right)}\right]^{-1}.$$

The sought distribution of the electric field in space, as well as the shape of the free surface of the liquid, can be found by solving a simple ordinary differential equation

$$d\Phi / dz = W(\Phi).$$

Integrating this equation, we finally obtain:

$$\boxed{z(\Phi) = -i\Phi + \frac{ip_S}{\sqrt{p_E}}\arctan\sqrt{p_E + (p_E + 1)\tan^2(B\Phi / p_S)},}$$

This solution (and, hence, the relation $E = \sin\theta$) is compatible with the boundary conditions for

$$e\left(p_E\right) = 1 + \sqrt{p_E + 1}, \qquad d\left(p_E\right) = \pi\left(1 + \sqrt{p_E + 1}\right) \Big/ \operatorname{arsinh}\left(\sqrt{p_E}\right),$$

$$p_S\left(p_E\right) = \sqrt{p_E\left(p_E + 1\right)} \Big/ \operatorname{arsinh}\left(\sqrt{p_E}\right), \qquad B = \sqrt{p_E(p_E + 1)}.$$

Exact solution

The equilibrium shape of the jet is given by the following expression:

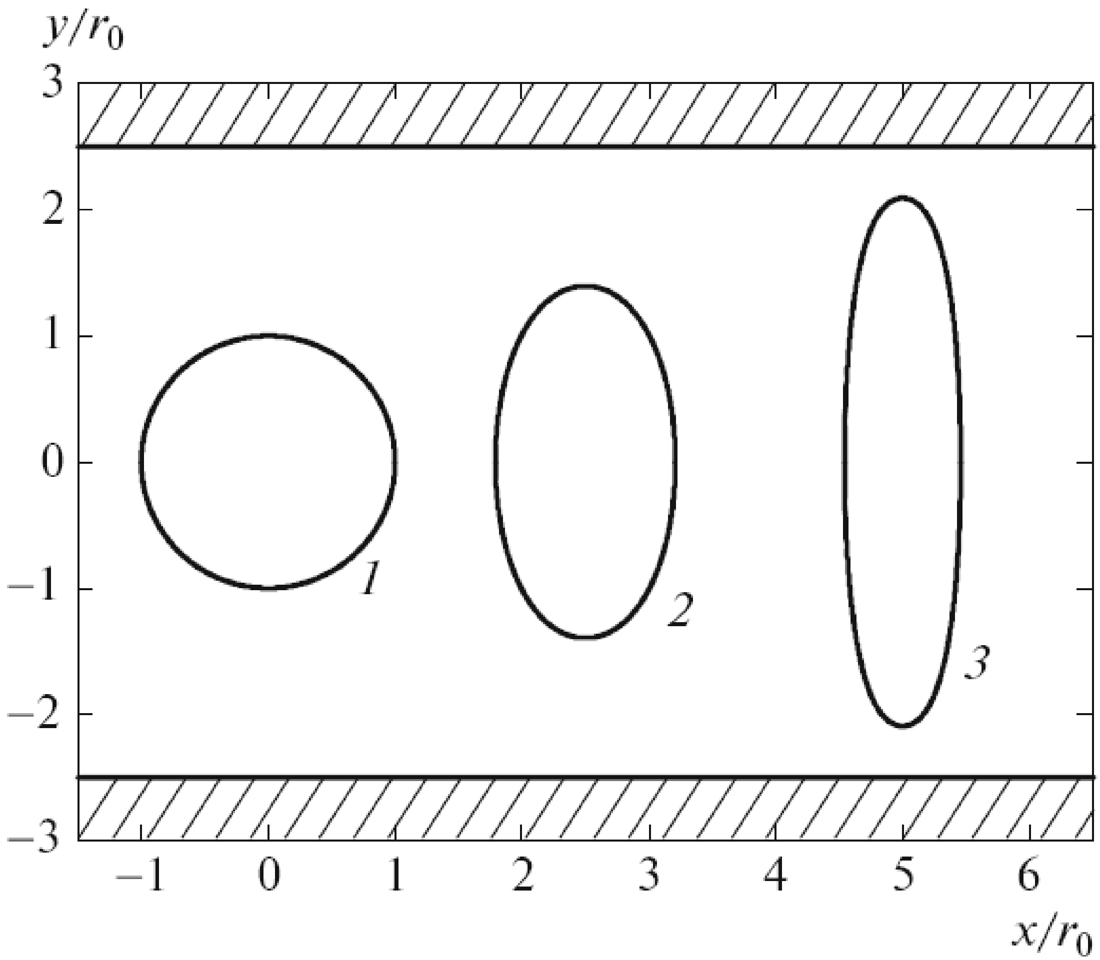
$$\boxed{y = \pm \frac{p_S}{\sqrt{p_E}} \arctan\sqrt{p_E - (p_E + 1)\operatorname{th}^2(Bx / p_S)}, \qquad -1 \le x \le 1.}$$

The obtained solution depend on a single control parameter p_E .

The cross-section area of the jet is given by the formula

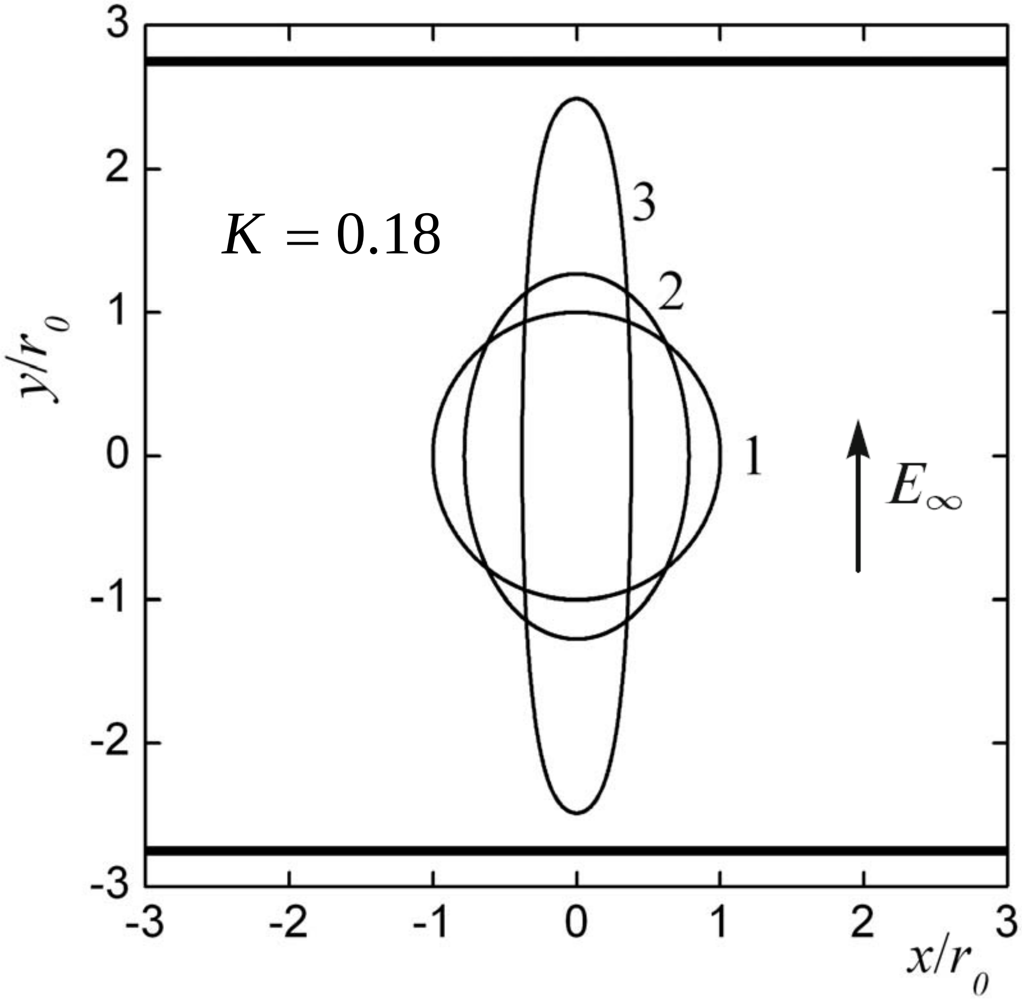
$$s(p_E) = \frac{\pi \ln(p_E + 1)\sqrt{p_E + 1}}{\operatorname{arsinh}^2\left(\sqrt{p_E}\right)}.$$

Then the radius of the jet is the following: $r_0 = \sqrt{s / \pi}.$



The size of the jet in the y direction is defined by

$$h\left(p_E\right) = \arctan\left(\sqrt{p_E}\right)\sqrt{1+p_E} \Big/ \operatorname{arsinh}\left(\sqrt{p_E}\right).$$



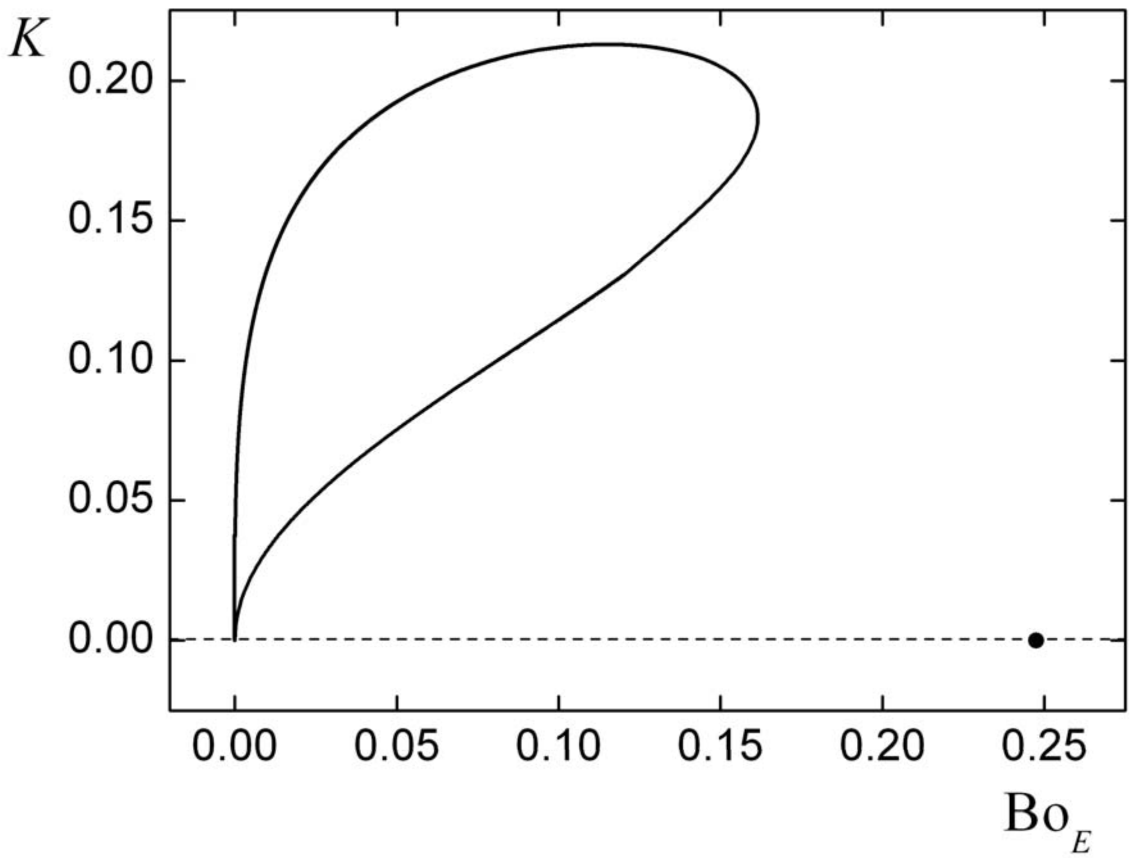
The cross section of the jet is unboundedly stretched in the direction of the applied field: $h/d \to 1$ as $e \to \infty$.

It is clear from general considerations that the equilibrium configuration of a jet of given radius R_0 is defined by two parameters, the interelectrode distance D and the potential difference U.

It is convenient to introduce (instead of D and U) two dimensionless parameters, the ratio of the scales K and the electric Bond number Bo_E:

$$K = R_0/D, \qquad \mathrm{Bo}_E = \varepsilon_0 R_0 E_\infty^2/(2T).$$

The values of the parameters K and Bo_E for the obtained exact solutions are shown in the figure.



The point

$$K = 0, \quad \mathrm{Bo}_E \approx 0.247$$

corresponds to the only previously known exact solution of the problem considered.

Approximate solutions; two-point method

Let us construct the general two-parameter family of solutions using the following approximation for the shape of a boundary:

$$y/x_{\max} = \pm c \cdot \sqrt{a+1} \cdot \arctan \sqrt{a - (a+1) \cdot \tanh^2 \left(x \cdot \operatorname{arsinh}\sqrt{a} / x_{\max} \right)} \Big/ \operatorname{arsinh}\sqrt{a}\,.$$

where a and c are the parameters which define the shape of the jet. This formula gives the exact one-parameter solution (with the parameter a) in the particular case where c = 1.

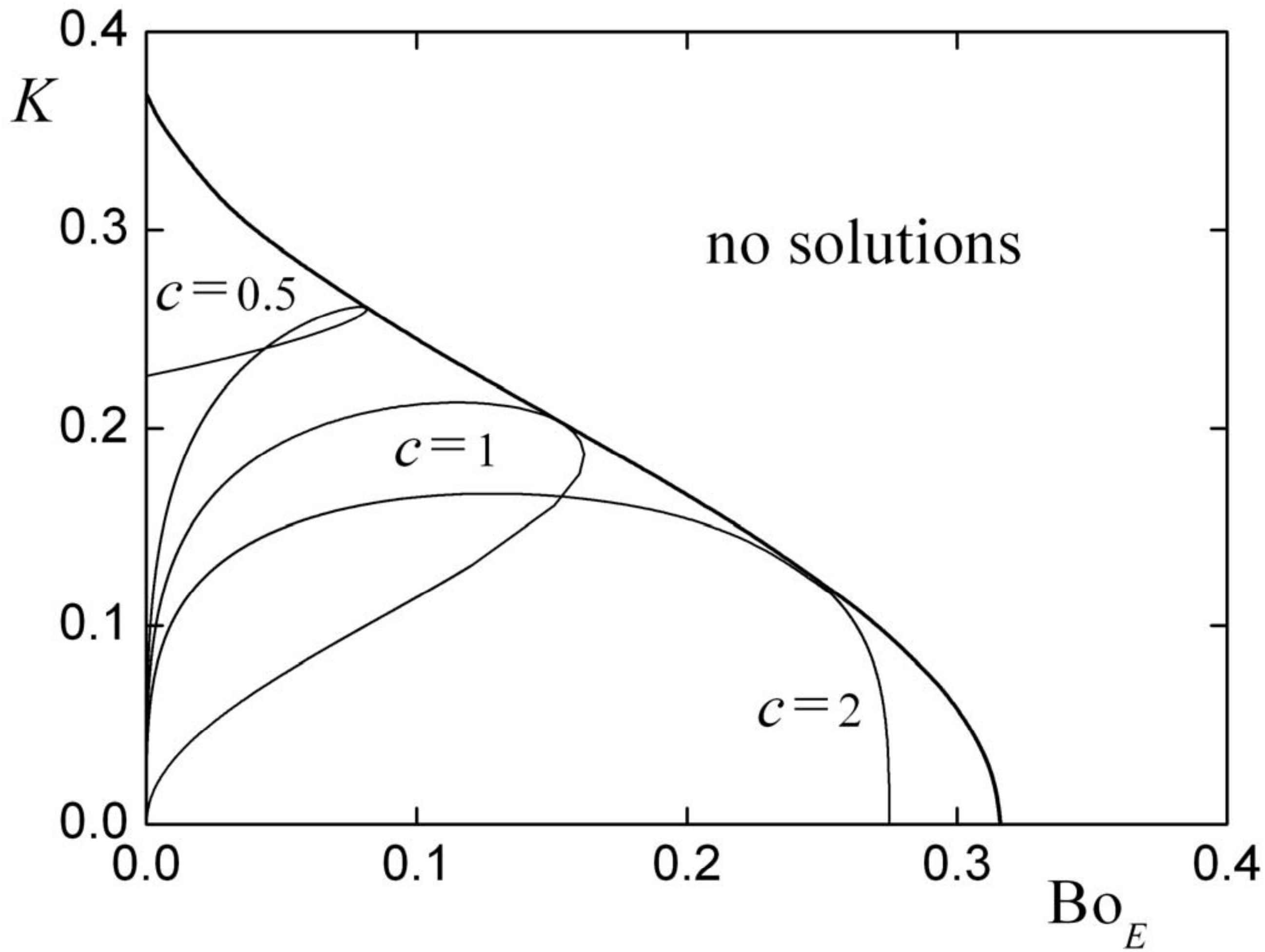
The field distribution in the interelectrode space is given by

$$z(w) = -iw/E_{\max} + icx_{\max}\sqrt{a+1}\,\operatorname{atan}\sqrt{a + (a+1)\tan^2\left(w \operatorname{arsinh}\sqrt{a} / (E_{\max} x_{\max})\right)} \Big/ \operatorname{arsinh}\sqrt{a}\,.$$

We require that the force balance condition be satisfied only at the points $\{0, \pm y_{\max}\}$, $\{\pm x_{\max}, 0\}$. As a result, we obtain the relation between the solution parameters, c and a, and the problem parameters, Bo_E and K :

$$\mathrm{Bo}_E = \frac{\left(c^3 (a+1) - 1\right)}{c^3 \left(1 + c\sqrt{a+1}\right)^2} \sqrt{\frac{c \ln(a+1)}{a\sqrt{a+1}}}, \qquad K = \frac{\sqrt{c \ln(a+1)\sqrt{a+1}}}{\pi\left(1 + \sqrt{a+1}\right)}.$$

The domain of existence of the solutions is shown. Also, the families of solutions for $c = 0.5, 1, 2$ are presented. The corresponding curves are intersected. So, two different solutions with different degrees of deformation coexist for each Bo_E and K.



Let us determine which of two solutions is realized. To this end, we will estimate the free energy of the system.

The energy analysis

The change of the total energy of the system (per unit length) before and after introduction of the jet into the interelectrode space is the following:

$$\boxed{\Delta W = \Delta W_L + \Delta W_E = TL - \varepsilon_0 E_\infty^2 ks/2.}$$

where $k = 1 + c\sqrt{a+1}$ and $s = \pi c \ln(a+1)\sqrt{a+1}/\operatorname{arsinh}^2\sqrt{a}$. The first term corresponds to the energy of the surface tension and the second term is responsible for the electric field energy. The perimeter L of the jet is calculated using the formula

$$L = \frac{\pi x_{\max}\left(61h^4 + 268h^3 + 366h^2 + 268h + 61\right)}{16(h+1)(h+3)(3h+1)}, \qquad h = \frac{\arctan\left(\sqrt{a}\right) c\sqrt{a+1}}{\operatorname{arsinh}\left(\sqrt{a}\right)}.$$

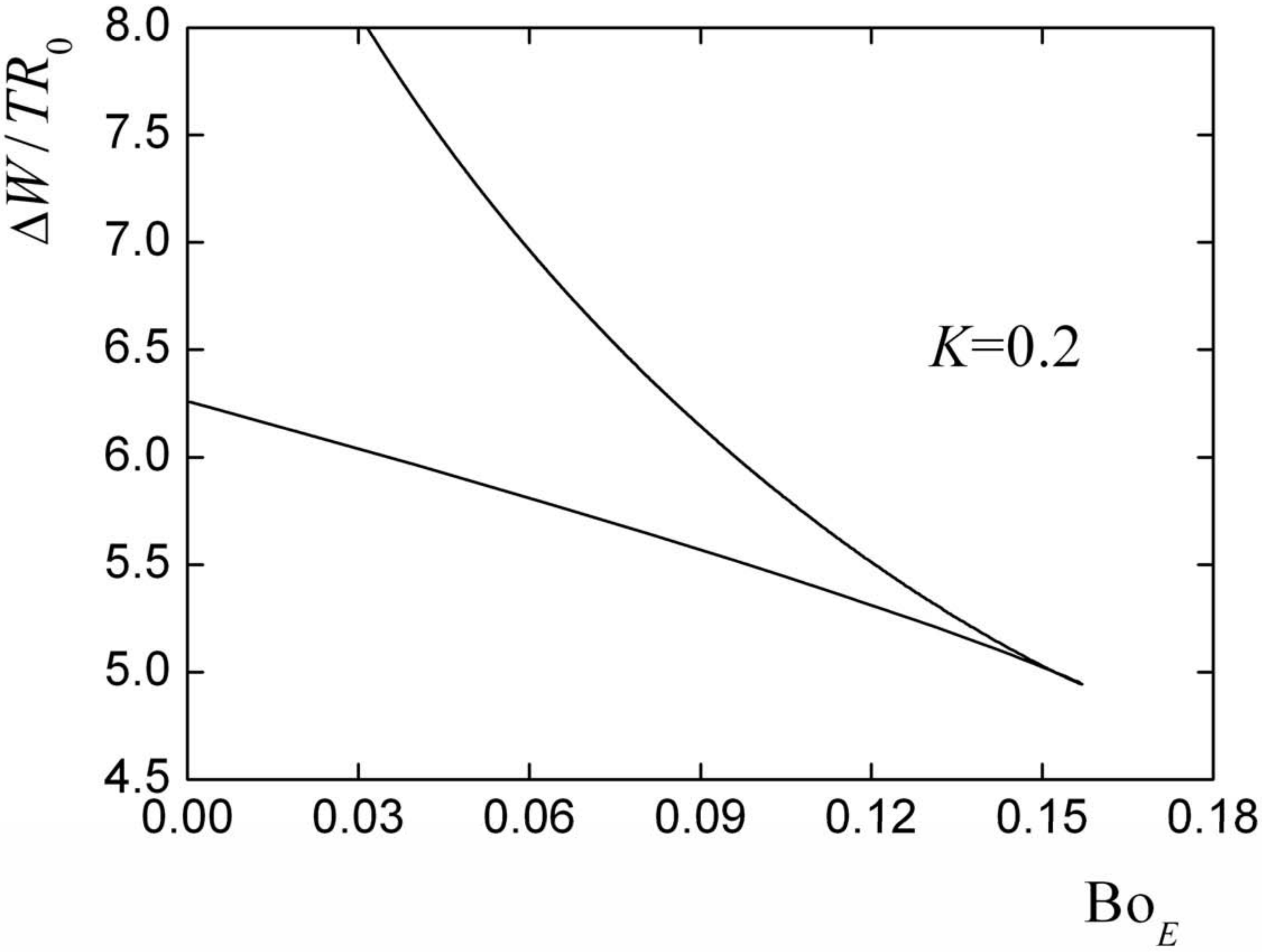
The change of the electric field energy is defined by

$$\Delta W_E = \frac{\varepsilon_0 E_{\max}^2 x_{\max}^2}{2}\left(\int_V \left(E^2 - \frac{1}{k^2}\right) dxdy - \frac{s}{k^2}\right).$$

In the conformal variables, this expression can be rewritten as

$$\Delta W_E = \frac{4}{k^2}\int_0^\infty \int_{-d/(2k)}^0 \left[1 - \frac{1}{\left|E(w)\right|^2 k^2}\right] d\varphi\, d\psi - \frac{s}{k^2} = \frac{\varepsilon_0 E_{\max}^2 x_{\max}^2 \pi c\sqrt{a+1}\ln(a+1)}{2\left(1 + c\sqrt{a+1}\right)\operatorname{arsinh}^2\left(\sqrt{a}\right)}.$$

The dependence of the change of free energy ΔW on the electric Bond number Bo_E for $K = 0.2$. Two different solutions with different degrees of deformation can coexist. The solution with lower energy is stable. It corresponds to a less deformed jet.



Conclusion

One-parameter family of exact solutions is obtained for the problem of an equilibrium configuration of an uncharged cylindrical jet of a conducting liquid in a transverse electric field [1]. The cross section of the jet is significantly (formally, unboundedly) stretched along the lines of forces of the field, and the boundaries of the jet asymptotically approach the electrodes. The only previously known, zero-parameter solution of the problem [2] does not belong to the obtained family of solutions.

Using the approximate solutions for the stationary shape of the jet, we find the range of the parameters (the applied potential difference and the interelectrode distance), where the problem of finding the equilibrium configurations of the jet has solutions. Also we obtain the conditions under which the solutions do not exist and, consequently, the jet splits.

[1]. N.B. Volkov, N.M. Zubarev, O.V. Zubareva, *JETP*, **122**, 950-955 (2016).

[2]. M.J. Miksis, *Phys. Fluids*, 1981, **24**, 1967.

Thank you
for attention!