Elastic interaction between 3D nonlinear waves on the surface of dielectric liquids in a horizontal electric field

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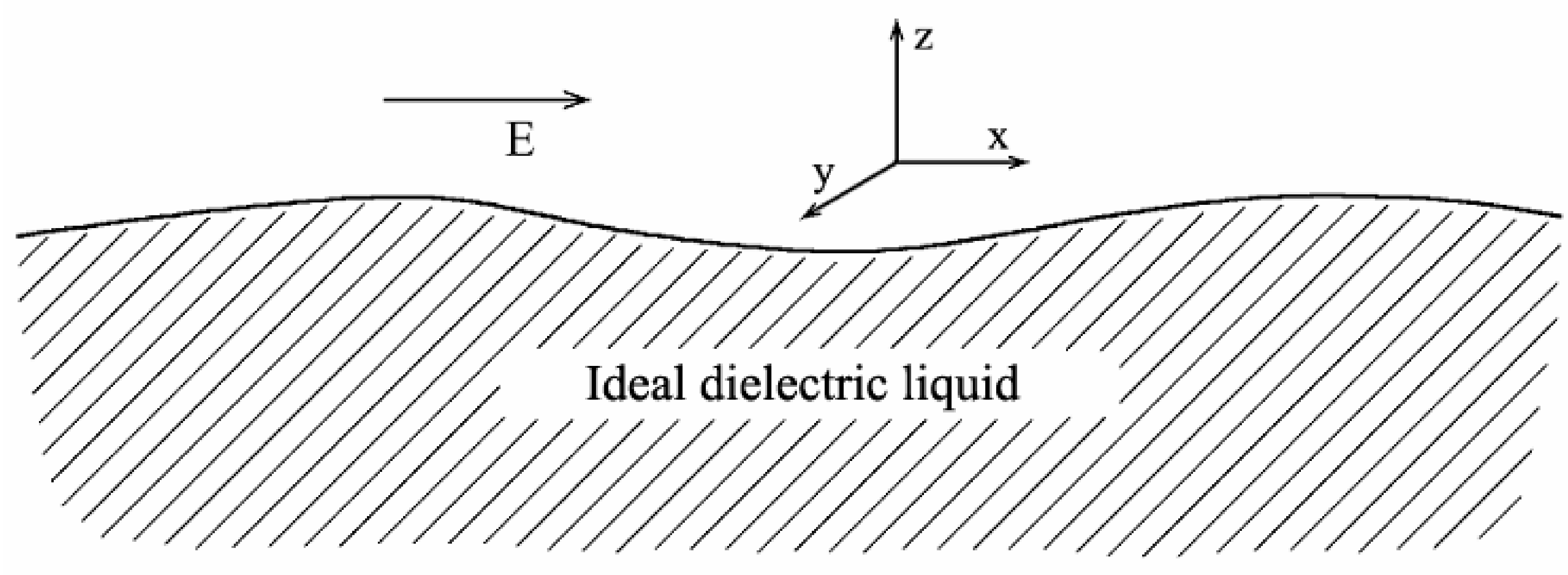
Previously, we have announced the talk with another title: "Nonlinear waves at the interface between two ideal fluids in a horizontal electric field in the presence of tangential discontinuity of the velocity".

Mainly, this work was published in Ref. [E.A. Kochurin, N.M. Zubarev, JETP Letters, 2016, **104**, 275].

We consider an ideal dielectric liquid of infinite depth with a free surface in a horizontal external electric field. In the unperturbed state, the liquid boundary is the horizontal plane, $z = 0$. Let the electric field vector be directed along the x axis and the function $\eta(x,y,t)$ be the deviation of the boundary from the plane; i.e., the region occupied by the liquid is confined by the free surface $z = \eta$.



Equations of motion

$$\nabla^2 \Phi = 0,$$

$$\rho \left(\frac{\partial \Phi}{\partial t} + \frac{(\nabla \Phi)^2}{2} \right) = P_E - P_0, \qquad z = \eta(x, y, t),$$

$$\frac{\partial \eta}{\partial t} = \frac{\partial \Phi}{\partial z} - \nabla_\perp \eta \cdot \nabla_\perp \Phi, \qquad z = \eta(x, y, t),$$

$$\Phi \to 0, \qquad z \to -\infty,$$

where Φ is the velocity potential,

ρ is the mass density,

P_E is the electrostatic pressure,

P_0 is a constant,

and $\nabla = \{\partial_x, \partial_y, \partial_z\}, \quad \nabla_\perp = \{\partial_x, \partial_y, 0\}.$

The electric field potential in the liquid satisfies the Laplace equation

$$\nabla^2 \varphi = 0.$$

It should be solved together with the following condition at infinity:

$$\varphi \to -Ex, \qquad z \to -\infty.$$

Let us consider the limiting case of high permittivity of the liquid, $\varepsilon >> 1$ (for example, $\varepsilon \approx$ 26, 36, and 81 for ethyl alcohol, nitrobenzene, and water, respectively). In this limit we have:

$$\frac{\partial \varphi}{\partial z} - \nabla_\perp \eta \cdot \nabla_\perp \varphi = 0, \qquad z = \eta(x, y, t),$$

i.e., the electric field lines are directed along the tangent to the curved surface. Then we find for the electrostatic pressure:

$$P_E \approx \varepsilon (\nabla \varphi)^2 / 8\pi, \qquad P_0 \approx \varepsilon E^2 / 8\pi, \qquad z = \eta(x, y, t).$$

As a consequence, the surface evolution is defined only by the electric field in the liquid.

Integrals of motion

For further consideration, it is important that the total energy W of the system and the momentum P along the x axis are integrals of motion. These quantities are given by the expressions:

$$W = \int_{z\le\eta} \left[\frac{\rho(\nabla\Phi)^2}{2} - \frac{\varepsilon(\nabla\varphi)^2}{8\pi} + \frac{\varepsilon E^2}{8\pi} \right] d^3r = \text{const},$$

$$P = \rho \int_{z\le\eta} \frac{\partial\Phi}{\partial x} d^3r = \text{const}.$$

We have added the last term into the integrand of the energy W in order to provide its finiteness. It is possible because of the mass conservation law:

$$M = \rho \int_{z\le\eta} d^3r = \text{const}.$$

[E.A. Kuznetsov, M.D. Spektor, Sov. Phys. JETP **44**, 136 (1976)]

Exact particular solutions

The equations of motion admit a pair of exact partial 3D solutions,

$$\eta = \eta^{\pm}(x \mp ct, y), \qquad \Phi = \pm F^{\pm}(x \mp ct, y, z), \qquad \varphi = kF^{\pm}(x \mp ct, y, z) + Ex,$$

where the functions $\eta^{\pm}$ are arbitrary, and the functions $F^{\pm}$ are defined by the equations:

$$\nabla^2 F^{\pm} = 0,$$

$$-c\frac{\partial \eta^{\pm}}{\partial x} = \frac{\partial F^{\pm}}{\partial z} - \nabla_{\perp}\eta^{\pm} \cdot \nabla_{\perp} F^{\pm}, \qquad z = \eta^{\pm},$$

$$F^{\pm} \to 0, \qquad z \to -\infty.$$

These solutions correspond to waves of arbitrary geometry that propagate at a constant velocity $c = E/k > 0$ (here $k = (4\pi\rho/\varepsilon)^{1/2}$) without distortions along (upper signs) or against (lower signs) the external electric field [N.M. Zubarev, JETP Letters, 2009, **89**, 271].

Integrals of motion for the traveling wave solutions

Substituting these solutions into the integrals W and P, we find:

$$W = -c\rho \int\limits_{z \le \eta^{\pm}} \frac{\partial F^{\pm}}{\partial x} d^3 r, \qquad P = \pm\rho \int\limits_{z \le \eta^{\pm}} \frac{\partial F^{\pm}}{\partial x} d^3 r.$$

It follows from these expressions that the energy W and the momentum P are related by the following simple formula:

$$W = \mp cP.$$

Interaction of counterpropagating solitary waves

Let $\eta \to 0$ as $x^2 + y^2 \to \infty.$

Before the collision of the waves, i.e. for $t \to -\infty,$ we have:

$$\eta(x,y,t) = \eta_1^+(x-ct,y) + \eta_1^-(x+ct,y),$$

$$\Phi(x,y,z,t) = F_1^+(x-ct,y,z) - F_1^-(x+ct,y,z),$$

$$\varphi(x,y,z,t) = kF_1^+(x-ct,y,z) + kF_1^-(x+ct,y,z) + Ex.$$

After the collision, for $t \to +\infty$ we have:

$$\eta(x,y,t) = \eta_2^+(x-ct,y) + \eta_2^-(x+ct,y),$$

$$\Phi(x,y,z,t) = F_2^+(x-ct,y,z) - F_2^-(x+ct,y,z),$$

$$\varphi(x,y,z,t) = kF_2^+(x-ct,y,z) + kF_2^-(x+ct,y,z) + Ex.$$

Here the subscripts “1” and “2” correspond to the waves before and after their collision. All the functions $F_{1,2}^{\pm},\ \eta_{1,2}^{\pm}$ are spatially localized.

Since the waves are spatially localized, it is possible to divide the energy W and the momentum P into the parts, corresponding to the counterpropagating waves. Before the collision ($t \to -\infty$), we have:

$$W = W_1^+ + W_1^-, \qquad W_1^\pm = -c\rho \int\limits_{z \le \eta_1^\pm} \frac{\partial F_1^\pm}{\partial x} d^3 r,$$

$$P = P_1^+ + P_1^-, \qquad P_1^\pm = \pm\rho \int\limits_{z \le \eta_1^\pm} \frac{\partial F_1^\pm}{\partial x} d^3 r.$$

After the collision ($t \to +\infty$), we have:

$$W = W_2^+ + W_2^-, \qquad W_2^\pm = -c\rho \int\limits_{z \le \eta_2^\pm} \frac{\partial F_2^\pm}{\partial x} d^3 r,$$

$$P = P_2^+ + P_2^-, \qquad P_2^\pm = \pm\rho \int\limits_{z \le \eta_2^\pm} \frac{\partial F_2^\pm}{\partial x} d^3 r.$$

As it was shown before, the following relations between the energies $W^\pm$ and the momenta $P^\pm$ are realized:

$$W^+ = -cP^+, \qquad W^- = +cP^-.$$

The energy and momentum conservation laws take the following form:

$$W_1^+ + W_1^- = W_2^+ + W_2^-,$$
$$P_1^+ + P_1^- = P_2^+ + P_2^-.$$

Taking into account the relations between the energy and momentum for the waves propagating in different directions, we get:

$$-cP_1^+ + cP_1^- = -cP_2^+ + cP_2^-,$$
$$P_1^+ + P_1^- = P_2^+ + P_2^-.$$

Combining these two equations, we can find that

$$P_1^+ = P_2^+, \qquad P_1^- = P_2^-,$$
$$W_1^+ = W_2^+, \qquad W_1^- = W_2^-.$$

This means that the energy and momentum of oppositely propagating solitary waves of arbitrary geometry are conserved at collision; i.e., the interaction between them is elastic.

The energies and momenta of individual waves for $t \to \pm\infty$ can be represented as combinations of the invariants W and P.

$$W^{\pm} = \frac{W \mp cP}{2}, \qquad P^{\pm} = \frac{\mp W + cP}{2c}.$$

It is noteworthy that this situation is in many respects similar to Alfven waves in an ideal liquid. Wave packets with an arbitrary shape can propagate without distortions either along or against the direction of the external magnetic field. The interaction is possible only between oppositely propagating waves and is elastic similar to the problem under consideration. The wave packets emerge from a collision with the same energy they had before the collision. This remarkable conservation law is related (see e.g. [Moffatt, 1978]) to the conservation of the *cross helicity* defined as

$$H = \frac{1}{2}\int (\vec{V} \cdot \vec{B})\, d^3 r.$$

Conclusion:

The nonlinear dynamics of the free surface of an ideal dielectric liquid with a large relative permittivity in a strong horizontal electric field has been considered. It has been demonstrated that the interaction between oppositely propagating solitary 3D waves of arbitrary geometry is elastic: they conserve their energy and momentum.

Thank you

for attention!