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Lecture notes

Conformal geometry and Riemann surfaces

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Lecture 1

Introduction to conformal geometry

Conformal geometry naturally arises when one deals with the phase transitions of the second order. When one approaches the phase transition point, the correlation length increases, we have scale invariance. Polyakov stated at critical point one the theory is not only scale invariant, but it is also conformal invariant. The conformal invariance allows to calculate explicitly critical indexes using representation theory for infinite-dimension Lie algebras.

Definition 1. *Let M^n be a smooth real manifold. Two Riemannian metrics $g_{ij}(x)$, $h_{ij}(x)$ belong to the same conformal class if $g_{ij}(x) = \lambda(x)g_{ij}(x)$ where $\lambda(x) > 0$ is some scalar function.*

Definition 2. *Let M^n be a smooth real manifold.* ***Conformal structure*** *on M^n is a fixed conformal class of Riemannian metrices on M^n.*

Of course, one can define a conformal structure on a smooth manifold by defining a Riemannian metric on it, but infinitely many Riemannian metrics define the same conformal structure on it, and there is no canonical choice of such metric.

Definition 3. *Let $(M^n, g_{ij}(x))$, $(N^n, h_{ij}(x))$ be 2 smooth Riemannian manifolds. A map $f : M^n \to N^n$ is called conformal if $g_{ij}(x) = \lambda(x)(f_*h)_{ij}(x)$. Here*

$$(f_*h)_{ij}(x) = \frac{\partial y_k}{\partial x_i}\frac{\partial y_l}{\partial x_j}h_{kl}(y(x))$$

By analogy with conformal mapping of Rimannian manifolds, one can define conformal mappings of conformal manifolds.

Definition 4. *Let M^n, N^n be a pair of smooth manifolds equipped with conformal structures, $g_{ij}(x)$, $h_{ij}(x)$ be* ***some*** *Riemannian metrices, representing the conformal structures on M^n, N^n, respectively. A map $f : M^n \to N^n$ is said to be conformal if $g_{ij}(x) = \lambda(x)(f_*h)_{ij}(x)$.*

It is clear that this definition is correct, i.e. it does not depend on the choice of the metrics within they conformal classes.

Lemma 1. *A map $f : M^n \to N^n$ is called conformal, if it preserves the angles between two curves.*

Let us now discuss the conformal maps in different dimensions.

1) For $n = 1$ the situation is trivial: any regular map is conformal.

2) For $n = 2$ we can treat the real plane $\mathbb{R}^2$ as the complex plane $\mathbb{C}^1$. If a map

$f : U \subset \mathbb{C} \to V \subset \mathbb{C}$ is conformal and preserve the orientation, then $f(z)$ is holomorphic. Hence, the space of local conformal maps is infinite dimensional. The global conformal maps of Riemann sphere $S^2 = \mathbb{C}P^1$ form the Möbius group.

3) For $n > 2$, the space of **local** conformal maps is finite-dimensional. For conformally flat manifolds is coincides with the Möbius group. Equivalently, any conformal map $f : U \to V$ from an open set $U \subset \mathbb{R}^n$ to an open set $V \subset \mathbb{R}^n$ is the restriction of some Möbius map $f : \mathbb{R}^n \to \mathbb{R}^n$ to U.

Let us consider 2-dimensional sphere $S^2 = \mathbb{C}P^1$. We have the following basic Möbius maps:

1) Shifts $z \to z + a$,
2) rotations $z \to e^{i\varphi} z,\ \varphi \in \mathbb{Z}$,
3) dilations $z \to \lambda z,\ \lambda \in \mathbb{R},\ \lambda > 0$,
4) inversion: $z \to \frac{1}{z}$,
5) complex conjugation: $z \to \overline{z}$

The subgroup of Möbius group preserving the orientation coincides with the group of fractional-linear complex transformations

$$z \to \frac{az + b}{cz + d}, \quad \text{where} \quad ad - bc \neq 0.$$

This group can be also interpreted as the group of projective transformations of $\mathbb{C}P^1$.

The Möbius group is defined for arbitrary n. It is natural to consider elements of Möbius group as conformal maps from S^n to S^n. Elements of Möbius group can be written in the following form:

$$\vec{x} \to \vec{b} + \alpha \frac{A(\vec{x} - \vec{a})}{||\vec{x} - \vec{a}||^{\varepsilon}}, \qquad (1)$$

where $\vec{a}$, $\vec{b}$, are arbitrary vectors from $\mathbb{R}^n$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $\varepsilon = 2$ or $\varepsilon = 0$, $A \in O(n)$, i.e. A is orthogonal, but not necessary preserve the orientation.

For $\mathbb{R}^n$ we have the following basic Möbius transformations:

1) Translations $\vec{x} \to \vec{x} + \vec{a}$,
2) rotations $\vec{x} \to A\vec{x},\ A \in \mathrm{O}(n)$,
3) dilations $\vec{x} \to \lambda \vec{x}, \lambda \neq 0$,
4) inversions: $\vec{x} \to \frac{\vec{x}}{||\vec{x}||^2}$.
5) reflections: $\vec{x} \to \vec{x} - 2(\vec{x}, \vec{n})\ \vec{n},\ ||\vec{n}|| = 1$.