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Lecture notes

Conformal geometry and Riemann surfaces

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Lecture 12

Action of vector fields at the fermionic Dirac space. The Virasoro algebra as central extension of the Witt algebra

Assume that we have a compact Riemann surface Γ with a marked point P_0, local parameter z near P_0, $z(P_0) = 0$ and the contour γ: $|z| = 1$ surrounding P_0.

Consider the following basis of j- tensors near γ

$$f_n = z^n (dz)^j$$

and the Witt algebra of vector fields on γ

$$v_k = z^{k+1}\partial_z.$$

with the commutators

$$\begin{aligned} [v_k, v_l] &= (z^{k+1}\partial_z)\cdot(z^{l+1}\partial_z) - (z^{l+1}\partial_z)\cdot(z^{k+1}\partial_z) = \\ &= z^{k+l+2}\partial_z^2 + (l+1)z^{k+l+1}\partial_z - z^{k+l+2}\partial_z^2 + (k+1)z^{k+l+1}\partial_z = \\ &= (l-k)z^{k+l+1}\partial_z = (l-k)v_{k+l}. \end{aligned} \tag{1}$$

Denote by l_k, $k \in \mathbb{Z}$ the Lie derivative of the basic tensors with respect to the vector field v_k:

$$l_k f_n = L_{v_k} f_n,$$

therefore

$$l_k f_n := (n + j(k+1)) f_{k+n}. \tag{2}$$

Lemma 1. *Operators l_p form the Witt algebra with the commutation relations:*

$$[l_p, l_q] = (q-p) l_{p+q}, \quad p, q \in \mathbb{Z} \tag{3}$$

Proof.

$$\begin{aligned} &[l_p, l_q] f_n = l_p l_q f_n - l_q l_p f_n = l_p(n + j(q+1)) f_{n+q} - l_q(n + j(p+1)) f_{n+p} = \\ &= (n+q+j(p+1))(n+j(q+1)) f_{n+p+q} - (n+p+j(q+1))(n+j(p+1)) f_{n+p+q} = \\ &= \big[\cancel{n^2} + nq + \cancel{jn(q+1)} + jq(q+1) + \cancel{jn(p+1)} + \cancel{j^2(p+1)(q+1)} - \\ &- \cancel{n^2} - np - \cancel{jn(p+1)} - jp(p+1) - \cancel{jn(q+1)} - \cancel{j^2(q+1)(p+1)}\big] f_{n+p+q} = \\ &= \big[n(q-p) + j(q^2 - p^2 + q - p)\big] f_{n+p+q} = (q-p)\big[n + j(q+p+1)\big] f_{n+p+q} = (q-p) l_{q+p} f_n. \end{aligned}$$

□

The action can be naturally extended to finite wedge products of the basic elements using the Leibniz formula:

$$l_k(\omega_1 \wedge \omega_2) = (l_k \omega_1) \wedge \omega_2 + \omega_1 \wedge (l_k \omega_2). \tag{4}$$

Lemma 2. *Extension of the action (2) to finite wedge products respects the commutation relations.*

Proof.

$$
\begin{aligned}
& l_p\Big(l_q\big(\omega_1 \wedge \omega_2\big)\Big) - l_q\Big(l_p\big(\omega_1 \wedge \omega_2\big)\Big) = \\
& = l_p\big((l_q\omega_1) \wedge \omega_2 + \omega_1 \wedge (l_q\omega_2)\big) - l_q\big((l_p\omega_1) \wedge \omega_2 + \omega_1 \wedge (l_p\omega_2)\big) = \\
& = (l_pl_q\omega_1) \wedge \omega_2 + \cancel{(l_q\omega_1) \wedge (l_p\omega_2)} + \cancel{(l_p\omega_1) \wedge (l_q\omega_2)} + \omega_1 \wedge (l_pl_q\omega_2) - \\
& - (l_ql_p\omega_1) \wedge \omega_2 - \cancel{(l_p\omega_1) \wedge (l_q\omega_2)} - \cancel{(l_q\omega_1) \wedge (l_p\omega_2)} + \omega_1 \wedge (l_pl_q\omega_2) = \\
& = (l_pl_q\omega_1) \wedge \omega_2 + \omega_1 \wedge (l_pl_q\omega_2) - (l_ql_p\omega_1) \wedge \omega_2 - \omega_1 \wedge (l_ql_p\omega_2) = \\
& = ([l_p, l_q]\omega_1) \wedge \omega_2 + \omega_1 \wedge ([l_p, l_q]\omega_2) = [l_p, l_q]\big(\omega_1 \wedge \omega_2\big).
\end{aligned}
$$

□

Let us recall that the wedge product is skew-symmetric

$$f_j \wedge f_k = -f_k \wedge f_j.$$

In particular, if a form contains the same multiplier f_k more than once, it is equal to zero.

The Dirac-type fermionic Fock space is generated by the following "stable" semi-infinite wedge products:

$$\omega = f_{k_1} \wedge f_{k_2} \wedge \ldots \wedge f_{k_s} \wedge |N>,$$

where $|N>$ denotes the semi-infinite wedge product of all basic tensors f_k with $k \geq N$ sorted in the ascending order, ω denotes any **finite** wedge product of monomials.

$$|N>= f_N \wedge f_{N+1} \wedge f_{N+2} \wedge \ldots, \quad \omega = f_{j_1} \wedge f_{j_2} \wedge \ldots \wedge f_{j_s}.$$

In particular, the vacuum vector $|0>$ is defined by:

$$|0>= f_0 \wedge f_1 \wedge f_2 \wedge f_3 \wedge f_4 \wedge f_5 \wedge \ldots$$

Lemma 3. *The action of l_p with $p \neq 0$ on the semi-infinite forms is well-defined, moreover, if $p+q \neq 0$, the commutation relations remain unchanged.*

Proof. It is natural to define the action of the l_p on the basic elements by:

$$l_p(\omega \wedge |N>) = l_p(\omega) \wedge |N> + \omega \wedge (l_p \wedge |N>),$$

where

$$
\begin{aligned}
l_p|N> = \; & l_p(f_N) \wedge f_{N+1} \wedge f_{N+2} \wedge f_{N+3} \wedge \ldots + \\
& + f_N \wedge (l_pf_{N+1}) \wedge f_{N+2} \wedge f_{N+3} \wedge \ldots + \\
& + f_N \wedge f_{N+1} \wedge (l_pf_{N+2}) \wedge f_{N+3} \wedge \ldots + \\
& + f_N \wedge f_{N+1} \wedge f_{N+2} \wedge (l_pf_{N+3}) \wedge \ldots + \ldots = \\
& = (N + j(p+1))f_{N+p} \wedge f_{N+1} \wedge f_{N+2} \wedge f_{N+3} \wedge \ldots + \\
& + (N + 1 + j(p+1))f_N \wedge f_{N+p+1} \wedge f_{N+2} \wedge f_{N+3} \wedge \ldots + \\
& + (N + 2 + j(p+1))f_N \wedge f_{N+1} \wedge f_{N+p+2} \wedge f_{N+3} \wedge \ldots + \\
& + (N + 3 + j(p+1))f_N \wedge f_{N+1} \wedge f_{N+2} \wedge f_{N+p+3} \wedge \ldots + \ldots
\end{aligned}
\tag{5}
$$

We see that for $p > 0$ all summands in contain repeating multipliers, therefore $l_p|N >= 0$ for all $p > 0$. If $p < 0$, he first $-p$ summands in (5) are non-vanishing, and all other summands are equal to zero, therefore we have finite sums. Again, if we calculate the commutator $[l_p, l_q]$, $p + q \neq 0$ all sums are finite and the commutator is not affected.

□

The action of l_0 on the semi-infinite form is not defined, because it contains infinitely many non-zero terms, therefore it requires a regularization. We shall use the following one:

$$l_0 := \frac{1}{2}[l_{-1}, l_1]. \tag{6}$$

For finite products this regularization coincides with the original definition. Let us calculate the commutator on the semi-infinite forms.

Let $p > 0$. As we pointed out, $l_p|N >= 0$, therefore

$$l_{-p}l_p|N >= 0.$$

$$\begin{aligned} l_{-p}|N > = &(N + j(1-p)) f_{N-p} \wedge f_{N+1} \wedge f_{N+2} \wedge \ldots \wedge f_{N+p-1} \wedge |N + p > + \\ &+ (N + 1 + j(1-p)) f_N \wedge f_{N-p+1} \wedge f_{N+2} \wedge \ldots \wedge f_{N+p-1} \wedge |N + p > + \\ &+ (N + 2 + j(1-p)) f_N \wedge f_{N+1} \wedge f_{N-p+2} \wedge \ldots \wedge f_{N+p-1} \wedge |N + p > + \\ &\ldots \\ &+ (N + p - 1 + j(1-p)) f_N \wedge f_{N+1} \wedge f_{N+2} \wedge \ldots \wedge f_{N-1} \wedge |N + p > . \end{aligned} \tag{7}$$

We see that only first p terms in (7) do not contain repetitive factors, therefore we have a finite sum.

After applying l_p to each summand in (7) we obtain exactly one non-zero monomial:

$$\begin{aligned} l_p l_{-p}|N > = &(l_p l_{-p} f_N) \wedge f_{N+1} \wedge f_{N+2} \wedge \ldots \wedge f_{N+p-1} \wedge |N + p > + \\ &+ f_N \wedge (l_p l_{-p} f_{N+1}) \wedge f_{N+2} \wedge \ldots \wedge f_{N+p-1} \wedge |N + p > + \\ &+ f_N \wedge f_{N+1} \wedge (l_p l_{-p}) f_{N+2} \wedge \ldots \wedge f_{N+p-1} \wedge |N + p > + \\ &\ldots \\ &+ f_N \wedge f_{N+1} \wedge f_{N+2} \wedge \ldots \wedge (l_p l_{-p} f_{N+p-1}) \wedge |N + p > . \end{aligned} \tag{8}$$

Taking into account that

$$l_{-p} f_{N+s} = (N + s + j(1-p)) f_{N-p+s},$$

$$l_p f_{N-p+s} = (N + s - p + j(1+p)) f_{N+s},$$

we obtain

$$l_p l_{-p} f_{N+s} = (N + s - p + j(1+p))(N + s + j(1-p)) f_{N+s},$$

$$l_p l_{-p}|N >= \sum_{s=0}^{p-1} (N + s - p + j(1+p))(N + s + j(1-p))|N >,$$

and

$$
\begin{aligned}
&\sum_{s=0}^{p-1}(N+s-p+j(1+p))(N+s+j(1-p)) = \\
=&\sum_{s=0}^{p-1}\left(N^2+2Ns+s^2+2j(N+s)+j^2(1-p^2)-pN-ps-j(p-p^2)\right)
\end{aligned}
\tag{9}
$$

Taking into account that

$$
\sum_{s=0}^{p-1}1=p,\quad \sum_{s=0}^{p-1}s=\frac{p(p-1)}{2},\quad \sum_{s=0}^{p-1}s=\frac{p(p-1)(2p-1)}{6},
$$

we obtain

$$
\begin{aligned}
&\sum_{s=0}^{p-1}(N+s-p+j(1+p))(N+s+j(1-p)) = \\
=&N^2p+Np(p-1)+\frac{p(p-1)(2p-1)}{6}+2jNp+jp(p-1)+j^2p(1-p^2)- \\
-&p^2N-\frac{p^2(p-1)}{2}-j(p^2-p^3)= \\
=&(N^2-N+2jN)p+j^2(p-p^3)+j(p^3-p)-\frac{p(p-1)(p+1)}{6}= \\
=&(N^2-N+2jN)p-2(6j^2-j+1)\frac{(p^3-p)}{12}.
\end{aligned}
\tag{10}
$$

Let us denote

$$
c_j=-2(6j^2-6j+1),
$$

c_j is called **central charge.** Then

$$
[l_p,l_{-p}]|N>\left((N^2-N+2jN)p+c_j\frac{(p^3-p)}{12}\right)|N>. \tag{11}
$$

In particular,

$$
\frac{1}{2}[l_{-1},l_1]|N>=-\left(\frac{N^2-N+2jN}{2}\right)|N>, \tag{12}
$$

therefore

$$
[l_p,l_{-p}]|N>=-2p\,l_0\,|N>+c_j\frac{(p^3-p)}{12}|N>. \tag{13}
$$

$$
\begin{aligned}
[l_p,l_{-p}](\omega\wedge|N>) &= -2p(l_0\omega)\wedge\,|N>-2p\,\omega\wedge(l_0\,|N>)+c_j\frac{(p^3-p)}{12}\omega\wedge|N>= \\
&= -2pl_0\,(\omega\wedge|N>)+c_j\frac{(p^3-p)}{12}\,(\omega\wedge|N).
\end{aligned}
\tag{14}
$$

Finally we obtain:

Theorem 1. *The action of regularized generators l_p on the Dirac space generated by "stable" semi-infinite wedge products of basic tensors satisfy the Virasoro relations*

$$\begin{aligned} &[l_p, l_q] = (q-p)l_{p+q} + \frac{(p^3-p)}{12}\delta_{p+q}c_j, \quad p, q \in \mathbb{Z} \\ &[c_j, l_p] = 0, \quad \textit{for all} \ \ p \in \mathbb{Z}. \end{aligned} \tag{15}$$

where c_j is the operator of multiplication to following number, known as central charge.

$$c_j = -2(6j^2 - 6j + 1), \tag{16}$$

where j is the tensor weight of the basic forms.