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Lecture notes

Conformal geometry and Riemann surfaces

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Lecture 2

Lie derivative

At the previous Lecture we introduced the notion of conformal manifold.

Definition 1. *Let M^n be a smooth real manifold. Two Riemannian metrics $g_{ij}(x)$, $h_{ij}(x)$ belong to the same conformal class if $g_{ij}(x) = \lambda(x) g_{ij}(x)$ where $\lambda(x) > 0$ is some scalar function.*

Definition 2. *Let M^n be a smooth real manifold.* ***Conformal structure*** *on M^n is a fixed conformal class of Riemannian metrices on M^n.*

Of course, one can define a conformal structure on a smooth manifold by defining a Riemannian metric on it, but infinitely many Riemannian metrics define the same conformal structure on it, and there is no canonical choice of such metric.

One of important operation in differential geometry is the **covariant derivative** of tensor fields. But there is another way how to define differentiation of tensor fields with respect to vector fields, known as **Lie derivative**. Let us recall the definition of Lie derivative.

Let M^n be a smooth manifold. Consider an infinitesimal transformation $\vec{x} \to \widetilde{x} = \vec{F}(\vec{x})$:

$$\widetilde{x} = F(\vec{x}) = \vec{x} + \varepsilon \vec{v}(\vec{x}).$$

In is well-known that infinitesimal transformations are generated by vector fields.

For our calculations it is convenient to introduces two independent set of indexes: $\{i_1, \dots, i_p, j_1, \dots, j_q\}$ and $\{\widetilde{i}_1, \dots, \widetilde{i}_p, \widetilde{j}_1, \dots, \widetilde{j}_1\}$.

Let $T = T^{i_1,\dots,i_p}_{j_1,\dots,j_q}(\vec{x})$ be a tensor field on M^n.

Definition 3. *The* ***Lie derivative*** *$L_{\vec{v}}T$ of the tensor field T along the vector field v is defined by*

$$(L_{\vec{v}}T)^{i_1,\dots,i_p}_{j_1,\dots,j_q}(\vec{x}) = \lim_{\varepsilon \to 0} \frac{(\vec{F}_* T)^{i_1,\dots,i_p}_{j_1,\dots,j_q}(\vec{x}) - T^{i_1,\dots,i_p}_{j_1,\dots,j_q}(\vec{x})}{\varepsilon}, \tag{1}$$

where $\vec{F}_* T$ denotes the pullback of the tensor field T with respect to the map $\vec{F}$:

$$(\vec{F}_* T)^{i_1,\dots,i_p}_{j_1,\dots,j_q}(\vec{x}) = \frac{\partial x^{i_1}}{\partial \widetilde{x}^{\widetilde{i}_1}} \cdots \frac{\partial x^{i_p}}{\partial \widetilde{x}^{\widetilde{i}_p}} \cdot \frac{\partial \widetilde{x}^{\widetilde{j}_1}}{\partial x^{j_1}} \cdots \frac{\partial \widetilde{x}^{j_q}}{\partial x^{j_q}} \cdot T^{\widetilde{i}_1,\dots,\widetilde{i}_p}_{\widetilde{j}_1,\dots,\widetilde{j}_q}(\vec{F}(\vec{x})). \tag{2}$$

Formula (2) can be found in standard differential geometry textbooks.

Let us calculate $L_{\vec{v}}T$ explicitly.

Lemma 1.

$$(L_{\vec{v}}T)^{i_1,\dots,i_p}_{j_1,\dots,j_q} = v^\alpha \frac{\partial T^{i_1,\dots,i_p}_{j_1,\dots,j_q}}{\partial x^\alpha} - \sum_{k=1}^{p} \frac{\partial v^{i_k}}{\partial x_\alpha} T^{i_1,\dots,\overset{\alpha}{\not{i_k}},\dots,i_p}_{j_1,\dots,j_q} + \sum_{k=1}^{q} \frac{\partial v^\alpha}{\partial x^{j_k}} T^{i_1,\dots,i_p}_{j_1,\dots,\underset{\alpha}{\not{j_k}},\dots,j_q}. \tag{3}$$

Here the notations $T^{i_1,\dots \overset{\alpha}{\not{i_k}},\dots,i_p}_{j_1,\dots,j_q}$ *means that we put* α *to the upper position* k *instead of* i_k*, analogously,* $T^{i_1,\dots,i_p}_{j_1,\dots,\underset{}{\overset{\alpha}{\not{j_k}}},\dots,j_q}$ *means that we* α *to the lower position* k *instead of* j_k*. For example,*

$$T^{i_1,i_2,\overset{\alpha}{\not{i_3}},i_4}_{j_1,j_2,j_3} = T^{i_1,i_2,\alpha,i_4}_{j_1,j_2,j_3}; \quad \textit{here} \quad k = 3.$$

Proof. Let us calculate $(\vec{F}_* T)^{i_1,\dots,i_p}_{j_1,\dots,j_q}(\vec{x})$ discarding $O(\varepsilon^2)$ terms. Since $\widetilde{x}^i = x^i + \varepsilon v^i(\vec{x})$, we obtain:

$$\frac{\partial \widetilde{x}^{\widetilde{i}}}{\partial x^i} = \delta^{\widetilde{i}}_i + \varepsilon \frac{\partial v^{\widetilde{i}}}{\partial x^i},$$

$$\frac{\partial x^i}{\partial \widetilde{x}^{\widetilde{i}}} = \delta^i_{\widetilde{i}} - \varepsilon \frac{\partial v^i}{\partial x^{\widetilde{i}}} + O(\varepsilon^2),$$

and

$$T^{i_1,\dots,i_p}_{j_1,\dots,j_q}(\vec{x} + \varepsilon \vec{v}) = T^{i_1,\dots,i_p}_{j_1,\dots,j_q}(\vec{x}) + \varepsilon v^\alpha \frac{\partial T^{i_1,\dots,i_p}_{j_1,\dots,j_q}(\vec{x})}{\partial x^\alpha} + O(\varepsilon^2).$$

Therefore,

$$(\vec{F}_* T)^{i_1,\dots,i_p}_{j_1,\dots,j_q}(\vec{x}) =$$

$$= \left[\delta^{i_1}_{\widetilde{i}_1} - \varepsilon \frac{\partial v^{i_1}(\vec{x})}{\partial x^{\widetilde{i}_1}}\right] \cdot \ldots \cdot \left[\delta^{i_p}_{\widetilde{i}_p} - \varepsilon \frac{\partial v^{i_p}(\vec{x})}{\partial x^{\widetilde{i}_p}}\right] \cdot \left[\delta^{\widetilde{i}_1}_{j_1} + \varepsilon \frac{\partial v^{\widetilde{j}_1}(\vec{x})}{\partial x^{j_1}}\right] \cdot \ldots \cdot \left[\delta^{\widetilde{j}_q}_{j_q} + \varepsilon \frac{\partial v^{\widetilde{j}_q}(\vec{x})}{\partial x^{j_q}}\right] \times$$

$$\times \left[T^{\widetilde{i}_1,\dots,\widetilde{i}_p}_{\widetilde{j}_1,\dots,\widetilde{j}_q}(\vec{x}) + \varepsilon v^\alpha(\vec{x}) \frac{\partial T^{\widetilde{i}_1,\dots,\widetilde{i}_p}_{\widetilde{j}_1,\dots,\widetilde{j}_q}(\vec{x})}{\partial x^\alpha}\right] + O(\varepsilon^2) =$$

$$= T^{i_1,\dots,i_p}_{j_1,\dots,j_q}(\vec{x}) + \varepsilon \left[v^\alpha(\vec{x}) \frac{\partial T^{i_1,\dots,i_p}_{j_1,\dots,j_q}(\vec{x})}{\partial x^\alpha} - \frac{\partial v^{i_1}(\vec{x})}{\partial x^{\widetilde{i}_1}} T^{\widetilde{i}_1,\dots,i_p}_{j_1,\dots,j_q}(\vec{x}) - \ldots - \frac{\partial v^{i_p}(\vec{x})}{\partial x^{\widetilde{i}_p}} T^{i_1,\dots,\widetilde{i}_p}_{j_1,\dots,j_q}(\vec{x}) + \right.$$

$$\left. + \frac{\partial v^{\widetilde{j}_1}(\vec{x})}{\partial x^{j_1}} T^{i_1,\dots,i_p}_{\widetilde{j}_1,\dots,j_q}(\vec{x}) + \ldots + \frac{\partial v^{\widetilde{j}_q}(\vec{x})}{\partial x^{j_q}} T^{i_1,\dots,i_p}_{j_1,\dots,\widetilde{j}_q}(\vec{x})\right] + O(\varepsilon^2) =$$

$$= T^{i_1,\dots,i_p}_{j_1,\dots,j_q}(\vec{x}) + \varepsilon \left[v^\alpha(\vec{x}) \frac{\partial T^{i_1,\dots,i_p}_{j_1,\dots,j_q}(\vec{x})}{\partial x^\alpha} - \frac{\partial v^{i_1}(\vec{x})}{\partial x^\alpha} T^{\alpha,\dots,i_p}_{j_1,\dots,j_q}(\vec{x}) - \ldots - \frac{\partial v^{i_p}(\vec{x})}{\partial x^\alpha} T^{i_1,\dots,\alpha}_{j_1,\dots,j_q}(\vec{x}) + \right.$$

$$\left. + \frac{\partial v^\alpha(\vec{x})}{\partial x^{j_1}} T^{i_1,\dots,i_p}_{\alpha,\dots,j_q}(\vec{x}) + \ldots + \frac{\partial v^\alpha(\vec{x})}{\partial x^{j_q}} T^{i_1,\dots,i_p}_{j_1,\dots,\alpha}(\vec{x})\right] + O(\varepsilon^2) =$$

□

Remark 1. *For scalar functions the Lie derivative coincides with the ordinary derivative along the vector field* $\vec{v}$*:*

$$(L_{\vec{v}} f)(x) = v^\alpha(x) \frac{\partial f(x)}{\partial x^\alpha}.$$

Remark 2. *There are two* ***completely different*** *operators acting on tensor fields on manifolds:*

1) Covariant derivative $\nabla_{\vec{v}}$.

2) Lie derivative $L_{\vec{v}}$.

The main differences between them are the following:

1)
- *To define the covariant derivative $\nabla_{\vec{v}}$ it is necessary to fix an additional structure –* ***connection****, which can be described using Christoffel symbols. There is no canonical connection on a smooth manifold M^n, unless it is equipped some additional structure. For example, on Riemannian manifolds Levi-Civitta connection is canonically defined.*
- *The Lie derivative is well-defined on arbitrary smooth manifold; no additional structure is required to introduce it.*

2)
- *For a fixed tensor filed T and a fixed point x_0 the value of the covariant derivative $\nabla_{\vec{v}}T$ is completely determined by the value of $\vec{v}$ at this point; no derivatives of $\vec{v}$ appear in the formula for $\nabla_{\vec{v}}T$.*
- *The value of the Lie derivative $L_{\vec{v}}T$ at a point x_0 depends not only on the value of $\vec{v}(x)$ at x_0, but also at the first derivatives $\vec{v}(x)$ at the point x_0.*

The covariant derivative is defined in all standard differential geometry courses. In contrast, Lie derivative is missing in many good books.

Remark 3. *Lie derivative arises naturally in the field theory. Assume that we have a Lie group of spacial symmetries in our theory. A tensor field T is invariant with respect to the action of the group if the Lie derivative of $L_eT = 0$ for any field e from the Lie algebra of the symmetry group.*