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Lecture notes

Conformal geometry and Riemann surfaces

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Lecture 4

Möbius group and Lorentz group

Let us construct a realization of Möbius transformations as the Lorentz transformation. Consider now the sphere S^n and the space $\mathbb{R}^{n+1,1}$, which is a $n+2$-dimensional space with the pseudoeuclidean metric.

We consider two main examples:

1) We will do calculations for $n = 3$.

2) We draw the figures for $n = 1$.

Consider the space $\mathbb{R}^{4,1}$ with the coordinates $(x = x^1, y = x^2, z = x^3, w = x^4, t = x^0)$ equipped with the pseudo-euclidean metric

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2 - dt^2.$$

Denote by C the light cone in $\mathbb{R}^{4,1}$

$$x^2 + y^2 + z^2 + w^2 - t^2 = 0. \tag{1}$$

Denote by PC the set of lines lying in C and passing through the origin. Equivalently, PC can be defined is the following way:

By definition, the projective space $\mathbb{R}P^4$ is the set of all lines in $\mathbb{R}^{4,1} = \mathbb{R}^5$ passing through the origin. Then PC is the algebraic surface in $\mathbb{R}P^4$ defined by equation (1). Let us recall that algebraic varieties in $\mathbb{R}P^n$ are defined by homogeneous polynomial equations.

Lemma 1. *The space PC is isomorphic to S^3.*

Proof. To define a line P_1 it is sufficient to choose 2 point on it. The elements of PC are lines passing through the origin $(0,0,0,0,0)$, therefore to fix a point in PC it is sufficient to fix one more point $(x_1, y_1, z_1, w_1, t_1)$ different from the origin and satisfying (1). Two points $(x_1, y_1, z_1, w_1, t_1)$ and $(x_2, y_2, z_2, w_2, t_2)$ define the same line iff

$$(x_1, y_1, z_1, w_1, t_1) = \lambda(x_2, y_2, z_2, w_2, t_2), \quad \lambda \in \mathbb{R}, \quad \lambda \neq 0.$$

From (1) it follows that $t_1 \neq 0$, therefore we can canonically choose the a point defining the line:

$$X = \left(\frac{x_1}{t_1}, \frac{y_1}{t_1}, \frac{z_1}{t_1}, \frac{w_1}{t_1}, 1\right).$$

It is clear that the point X is exactly the intersection of the line P_1 with the hyperplane $t = 1$, and we have one-to-one correspondence between PC and the intersection of the cone C with the hyperplane $t = 1$. The points of this intersection are defined by:

$$\begin{cases} x^2 + y^2 + z^2 + w^2 - t^2 = 0, \\ t = 1, \end{cases} \tag{2}$$

therefore we obtain:

$$x^2 + y^2 + z^2 + w^2 = 1. \tag{3}$$

Equation (3) is exactly the equation of $S^3 \subset \mathbb{R}^4$.

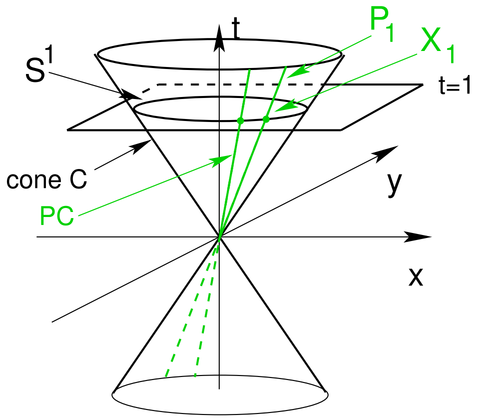

Figure 1. We draw the case $n = 1$. Here C is the light cone in $\mathbb{R}^{2,1}$, CP is the set of lines lying in the cone $C$and passing through the origin. The plane $t = 1$ intersects the cone by a circle S^1. Each line P from CP intersects S^1 at a point X, and through each point of S^1 passes exactly one line from PC. Therefore we have one-to-one correspondence between PC and S^1.

Let us check that the pseudo-Euclidean metric on $\mathbb{R}^{4,1}$ naturally defines conformal structure on PC, invariant with respect to the Lorentz group action. Let us denote the metric on $\mathbb{R}^{4,1}$ by $<,>$: if

$$\vec{u} = (u^1, u^2, u^3, u^4, u^0), \quad \vec{v} = (v^1, v^2, v^3, v^4, v^0),$$

then

$$< \vec{u}, \vec{v} >= u^1v^1 + u^2v^2 + u^3v^3 + u^4v^4 - u^0v^0$$

Remark 1. *Of course, the restriction of $<,>$ to the intersection of the cone C with the hyperplane $t = 1$ defines the standard Reimannian metric on S^3. But the the hyperplane $t = 1$ is not invariant under the action of the Lorentz group, therefore this Reimannian metric is also not inariant; in fact the the Lorentz group respects only its conformal class.*

Let $P_0 \in PC$ and X_0 be a point of the line P_0, different from the origin:

$$X_0 = (x_0, y_0, z_0, w_0, t_0), \quad < X, X >= 0.$$

Lemma 2. *The tangent space to PC at the point P_0 is isomorphic to the factor-space $T_{X_0}C/\{X_0\}$. Here $T_{X_0}C$ is the tangent space to the cone C at the point X_0 and $\{X_0\}$ denote the one-dimensional subspace generated by the vector X_0.*

Proof. Let P_1 be a line, infinitely close to P_0. Then it passes through a point

$$X_1 = X_0 + \varepsilon\vec{v} + o(\varepsilon).$$

Both points X_0, X_1 lie in the cone C, therefore v is a tangent vector to the cone C at the point X_0. But we have to take into account that, if we shift the point X_0 along the line P_0, the line remains the same. Therefore a vector $\vec{v}$ generates the trivial variation of the line iff

$$\vec{v} = \lambda X_0.$$

But it means exactly that we have to factor $T_{X_0}C$ by $\{X_0\}$. □

Let us recall that the cone C is defined by the equation:

$$F(X) = 0, \quad \text{where} \quad F(x) =< X, X >= x^2 + y^2 + z^2 + w^2 - t^2. \tag{4}$$

The tangent space $T_{X_0}C$ to the cone C at the point X_0 is defined by

$$(\nabla F(X)\big|_{X=X_0}, \vec{v}) = 0, \tag{5}$$

where $(,)$ is the standard pairing between vectors and covectors, and

$$\nabla F(X) = (2x, 2y, 2z, 2w, -2t).$$

Therefore equation (5) can be rewritten as:

$$< X_0, \vec{v} >= 0. \tag{6}$$

From (5) and (6) it follows immediately that $X_0 \in T_{X_0}C$.

Let us prove:

Lemma 3. *1) The restriction of the scalar product $<,>$ to the $T_{X_0}C$ is well-defined on $T_{X_0}C/\{X_0\}$.*

2) This restriction is positive defined.

Proof. To prove the first statement it is sufficient to check that if two pairs $\vec{u}_1, \vec{v}_1$ and $\vec{u}_2, \vec{v}_2$ represent the same pair of vectors $\tilde{u}, \tilde{v}$ from the factor-space, then

$$< \vec{u}_1, \vec{v}_1 >=< \vec{u}_2, \vec{v}_2 > .$$

But $\vec{u}_1$ and $\vec{u}_2$ represent the same vector in the factor-space iff

$$\vec{u}_2 = \vec{u}_1 + \alpha X_0,$$

analogously

$$\vec{v}_2 = \vec{v}_1 + \beta X_0.$$

We have

$$< \vec{u}_2, \vec{v}_2 >=< \vec{u}_1 + \alpha X_0, \vec{v}_1 + \beta X_0 >=< \vec{u}_1, \vec{v}_1 > + \alpha < \alpha X_0, \vec{v}_1 > + \beta < \vec{u}_1, X_0 > + \alpha\beta < X_0, X_0 >=$$

$$=< \vec{u}_1, \vec{v}_1 > .$$

Here we used formula (6). To prove the second part of the Lemma, let for calculating the scalar product on the factor-space we can freely choose representatives from the equivalence class. If

$$\vec{u} = (u^1, u^2, u^3, u^4, u^0), \quad \vec{v} = (v^1, v^2, v^3, v^4, v^0),$$

then

$$< u, v >=< \tilde{u}, \tilde{v} >,$$

where

$$\tilde{u} = \left(u^1 - \frac{u^0}{t_0} x_0, u^2 - \frac{u^0}{t_0} y_0, u^3 - \frac{u^0}{t_0} z_0, u^4 - \frac{u^0}{t_0} w_0, 0 \right)$$

$$\tilde{v} = \left(v^1 - \frac{v^0}{t_0} x_0, v^2 - \frac{v^0}{t_0} y_0, v^3 - \frac{v^0}{t_0} z_0, v^4 - \frac{v^0}{t_0} w_0, 0 \right)$$

But for the pairs $(\tilde{u}, \tilde{v})$ we have the standard Euclidean scalar product, restricted to a subspace, therefore it is non-degenerate and positive defined. □

Remark 2. *The scalar product defined in Lemma 3* ***does not define*** *Riemannian metric on PC, because this scalar product depends on the choice of X_0. But if we replace X_0 by λX_0, the scalar product is multiplied by λ^2. Therefore the Riemannian metric is defined up to a conformal factor, i.e. we have conformal structure.*

Taking into account that the Lorentz group preserves the scalar product $<, >$, we see that it preserves this conformal structure. Finally we come to the following conclusion:

Theorem 1. *Any Lorentz transformation of $\mathbb{R}^{4,1}$ generates a conformal transformation of S^3.*

It is easy to check that we obtain the full Möbius group.

Denote by A an element of the Lorentz group $O(4, 1)$:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,0} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,0} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,0} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,0} \\ a_{0,1} & a_{0,2} & a_{0,3} & a_{0,4} & a_{0,0} \end{bmatrix}$$

Denote by $O_+(4, 1)$ the subgroup of the Lorentz group defined by

$$a_{0,0} > 0.$$

It has two connected components, and $O_+(4, 1)$ is isomorphic to Möbius$(\mathbb{R}^3)$.

Let us provide explicit formulas for the action of $O_+(4, 1)$ at S^3. Consider the standard $S^3 \subset \mathbb{R}^4$ defined by

$$x^2 + y^2 + z^2 + w^2 = 1.$$

Then matrix $A \in O_+(4,1)$ defines the following map $S^3 \to S^3$:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \to \begin{bmatrix} \frac{a_{1,1}x+a_{1,2}y+a_{1,3}z+a_{1,4}w+a_{1,0}}{a_{0,1}x+a_{0,2}y+a_{0,3}z+a_{0,4}w+a_{0,0}} \\ \frac{a_{2,1}x+a_{2,2}y+a_{2,3}z+a_{2,4}w+a_{2,0}}{a_{0,1}x+a_{0,2}y+a_{0,3}z+a_{0,4}w+a_{0,0}} \\ \frac{a_{3,1}x+a_{3,2}y+a_{3,3}z+a_{3,4}w+a_{3,0}}{a_{0,1}x+a_{0,2}y+a_{0,3}z+a_{0,4}w+a_{0,0}} \\ \frac{a_{4,1}x+a_{4,2}y+a_{4,3}z+a_{4,4}w+a_{4,0}}{a_{0,1}x+a_{0,2}y+a_{0,3}z+a_{0,4}w+a_{0,0}} \end{bmatrix}.$$