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Lecture notes

Conformal geometry and Riemann surfaces

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Lecture 6

Weil and Cotton tensors

Definition 1. The Riemannian manifold $(M^n, g_{ij}(\vec{x}))$ is called flat, if near each point x_0 there exists a coordinate system $\tilde{x}^1, \dots, \tilde{x}^n$ such that in the new coordinates

$$\tilde{g}_{ij}(x) = \delta_{ij}. \quad (1)$$

is flat.

Definition 2. The conformal manifold $(M^n, \{g_{ij}(\vec{x})\})$ is called conformally flat, if near each point x_0 there exists a coordinate system $\tilde{x}^1, \dots, \tilde{x}^n$ such that in the new coordinates the Riemannian metric $g_{ij}(\vec{x})$ representing the conformal structure has the form

$$\tilde{g}_{ij}(x) = \lambda(x)\delta_{ij}. \quad (2)$$

is flat.

It is easy to check that the definition of conformal flatness does not depend on the choice of the Riemannian metric representing the conformal class.

Example 1. In the coordinates of the stereographic projection the natural metric on S^n has the form

$$ds^2 = 4 \frac{(dx^1)^2 + \dots + (dx^n)^2}{1 + (x^1)^2 + \dots + (x^n)^2},$$

therefore S^n equipped with the standard conformal structure is conformally flat. Analogously, Lobachevsky spaces L^n are conformally flat for all n :

$$ds^2 = 4 \frac{(dx^1)^2 + \dots + (dx^n)^2}{1 - (x^1)^2 + \dots - (x^n)^2},$$

Of course, neither S^n nor L^n are flat.

Let us recall some basic definitions from the Riemannian geometry.

1. Christoffel symbols:

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\alpha} [\partial_i g_{\alpha j} + \partial_j g_{i\alpha} - \partial_\alpha g_{ij}]$$

2. Riemann curvature tensor is defined by

$$(\nabla_k \nabla_l - \nabla_l \nabla_k) T^i = R^i{}_{jkl} T^j$$

Here T^i is a vector field.

Corollary:

$$(\nabla_k \nabla_l - \nabla_l \nabla_k) \alpha_q = -R^p{}_{qkl} \alpha_p,$$

where α_q is a one-form. Explicit expression:

$$R^i{}_{jkl} = \partial_k \Gamma_{lj}^i - \partial_l \Gamma_{kj}^i + \Gamma_{k\alpha}^i \Gamma_{lj}^\alpha - \Gamma_{l\alpha}^i \Gamma_{kj}^\alpha$$

By lowering the first index we obtain:

$$R_{ijkl} = g_{i\alpha} R^{\alpha}_{jkl}$$

It has the following symmetries:

- a) $R_{ijlk} = -R_{ijkl}$,
- b) $R_{jikl} = -R_{ijkl}$,
- c) $R_{klij} = R_{ijkl}$,

3. Biancki algebraic identities:

$$R_{sijk} + R_{sjki} + R_{skij} = 0.$$

Let us remark, that for $n = 2, 3$ these identities are trivial, and for $n \geq 4$ they are non-trivial.

4. Ricci tensor:

$$R_{kl} = R^{\alpha}_{k\alpha l} = g^{\alpha\beta} R_{\alpha k\beta l}$$

Ricci tensor is symmetric:

$$R_{kl} = R_{lk}.$$

5. Scalar curvature tensor:

$$R = g^{\alpha\beta} R_{\alpha\beta}$$

Let us recall that

Theorem 1. *A Riemannian manifold M^n with the metric g_{ij} admits flat coordinates such that $g_{ij} = \delta_{ij}$ iff the curvature tensor is identically zero:*

$$R^i_{jkl} \equiv 0, \quad \text{for all } i, j, k, l.$$

Definition 3. *A Riemannian manifold M^n with the metric g_{ij} is called **conformally flat** if there exists a scalar real function ω such that the Riemann metric $\tilde{g}_{ij} = e^{2\omega} g_{ij}$ is flat.*

How to check if a Riemannian manifold is conformally flat. The answer is provided by

Theorem 2. 1. *If $n = 2$ a metric g_{ij} is always conformally flat,*

2. *if $n = 3$ a metric is conformally flat iff its Cotton tensor is identically equal to zero: $C_{ikj} \equiv 0$;*

3. *if $n \geq 4$ a metric is conformally flat iff its Weyl tensor is identically equal to zero: $W_{ikjl} \equiv 0$.*

I do plan to provide a complete proof. The first part will be proved later in the 2-d part of the course. At the next lecture I plan to define these tensors and prove that for $n \geq 4$ the Weil tensor is conformally invariant, and the variation of the Cotton tensor with respect to conformal changes of metric is proportional to the Weyl tensor. If $n = 3$, the Weil tensor vanished identically, therefore the Cotton tensor is conformally invariant.

Let us define some basic objects

1. Schouten tensor:

$$P_{ij} = \frac{1}{n-2} \left[R_{ij} - \frac{1}{2(n-1)} R g_{ij} \right] \quad (3)$$

2. Weyl tensor:

$$W_{ijkl} = R_{ijkl} - [g_{ik}P_{jl} + g_{jl}P_{ik} - g_{il}P_{jk} - g_{jk}P_{il}] \quad (4)$$

$$W^i_{jkl} = R^i_{jkl} - [\delta^i_k P_{jl} + g_{jl} g^{i\alpha} P_{\alpha k} - \delta^i_l P_{jk} - g_{jk} g^{i\alpha} P_{\alpha l}] \quad (5)$$

3. Cotton tensor:

$$C_{jkl} = \nabla_l P_{jk} - \nabla_k P_{jl} \quad (6)$$

Let us also recall the standard definition:

$$\partial^k = g^{k\alpha} \partial_\alpha, \quad \nabla^k = g^{k\alpha} \nabla_\alpha.$$

We shall also use:

$$\nabla_k g_{ij} = 0.$$

Consider the following change of metric:

$$\tilde{g}_{ij} = e^{2\omega(x)} g_{ij} \quad (7)$$

Let us calculate step by step how this change of metric within the same conformal class affects the basic differential-geometric objects:

Theorem 3. *Let M^n , $g_{ij}(x)$ be a Riemannian manifold. If we introduce a new metric $\tilde{g}_{ij}(x)$ using formula (7), then we have the following transformation rules:*

- 1.

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + S^k_{ij},$$

where

$$S^k_{ij} = \delta^k_i \partial_j \omega + \delta^k_j \partial_i \omega - g_{ij} \partial^k \omega$$

- 2.

$$\begin{aligned} \tilde{R}^i_{jkl} &= R^i_{jkl} + \nabla_k S^i_{lj} - \nabla_l S^i_{kj} + S^i_{k\alpha} S^\alpha_{lj} - S^i_{l\alpha} S^\alpha_{kj} = \\ &= R^i_{jkl} + \delta^i_l \nabla_k \partial_j \omega - \delta^i_k \nabla_l \partial_j \omega + g_{jk} \nabla_l \partial^i \omega - g_{jl} \nabla_k \partial^i \omega + \\ &+ \delta^i_k \partial_j \omega \partial_l \omega - \delta^i_l \partial_j \omega \partial_k \omega + g_{jl} \partial^i \omega \partial_k \omega - g_{jk} \partial^i \omega \partial_l \omega + [\delta^i_l g_{jk} - \delta^i_k g_{jl}] [\partial^\alpha \omega \partial_\alpha \omega] \end{aligned}$$

3.

$$\tilde{R}_{jl} = R_{jl} - g_{jl}\nabla^\alpha\partial_\alpha\omega - (n-2)\nabla_l\partial_j\omega + (n-2)\partial_j\omega\partial_l\omega - (n-2)g_{jl}\partial^\alpha\omega\partial_\alpha\omega.$$

4.

$$\tilde{R} = e^{-2\omega}[R - 2(n-1)\nabla^\alpha\partial_\alpha\omega - (n-1)(n-2)\partial^\alpha\omega\partial_\alpha\omega]$$

Proof. 1. Let us now consider the change of metric in Christoffel symbols. Since

$$\partial_j\tilde{g}_{\alpha k} = 2[\partial_j\omega]e^{2\omega}g_{\alpha k} + e^{2\omega}\partial_jg_{\alpha k} = e^{2\omega}[\partial_jg_{\alpha k} + 2\partial_j\omega\tilde{g}_{\alpha k}]$$

we have

$$\tilde{\Gamma}_{jk}^i = \frac{1}{2}e^{-2\omega}g^{i\alpha}[e^{2\omega}\partial_jg_{\alpha k} + 2\partial_j\omega\tilde{g}_{\alpha k} + e^{2\omega}\partial_kg_{j\alpha} + 2\partial_k\omega\tilde{g}_{j\alpha} - e^{2\omega}\partial_\alpha g_{jk} - 2\partial_\alpha\omega\tilde{g}_{jk}]$$

and, similarly, taking into account the definition $\partial^k := g^{k\alpha}\partial_\alpha$

$$S_{jk}^i = g^{i\alpha}[\partial_j\omega\tilde{g}_{\alpha k} + \partial_k\omega\tilde{g}_{j\alpha} - \partial_\alpha\omega\tilde{g}_{\alpha k}] = \delta_k^i\partial_j\omega + \delta_j^i\partial_k\omega - \partial^i\omega g_{jk}.$$

2. For the Riemann tensor we obtain:

$$\begin{aligned}\tilde{R}_{jkl}^i &= \partial_k\Gamma_{lj}^i + \partial_kS_{lj}^i - \partial_l\Gamma_{kj}^i - \partial_lS_{kj}^i + [\Gamma_{k\alpha}^i + S_{k\alpha}^i][\Gamma_{lj}^\alpha + S_{lj}^\alpha] - [\Gamma_{l\alpha}^i + S_{l\alpha}^i][\Gamma_{kj}^\alpha + S_{kj}^\alpha] = \\ &= R_{jkl}^i + \partial_kS_{lj}^i - \partial_lS_{kj}^i + \Gamma_{k\alpha}^iS_{lj}^\alpha + \Gamma_{lj}^\alpha S_{k\alpha}^i + S_{k\alpha}^iS_{lj}^\alpha - \Gamma_{l\alpha}^iS_{kj}^\alpha - \Gamma_{kj}^\alpha S_{l\alpha}^i - S_{l\alpha}^iS_{kj}^\alpha - \\ &\quad - \Gamma_{kl}^\alpha S_{\alpha j}^i + \Gamma_{lk}^\alpha S_{lj}^i = R_{jkl}^i + \nabla_kS_{lj}^i - \nabla_lS_{kj}^i + S_{k\alpha}^iS_{lj}^\alpha - S_{l\alpha}^iS_{kj}^\alpha.\end{aligned}$$

3. For the Ricci tensor we obtain:

$$\tilde{R}_{jl} = \tilde{R}_{jil}^i = R_{jl} + \nabla_iS_{lj}^i - \nabla_lS_{ij}^i + S_{i\alpha}^iS_{lj}^\alpha - S_{l\alpha}^iS_{ij}^\alpha.$$

In order to compute it, we have to consider

$$\nabla_iS_{lj}^i - \nabla_lS_{ij}^i + S_{i\alpha}^iS_{lj}^\alpha - S_{l\alpha}^iS_{ij}^\alpha,$$

where

$$S_{i\alpha}^i = \delta_\alpha^i\partial_i\omega + \delta_i^i\partial_\alpha\omega - g_{\alpha i}\partial^i\omega = \partial_\alpha\omega + n\partial_\alpha\omega - \partial_\alpha\omega = n\partial_\alpha\omega.$$

So, we get

$$\begin{aligned}\nabla_iS_{lj}^i - \nabla_lS_{ij}^i &= \nabla_i[\delta_l^i\partial_j\omega + \delta_j^i\partial_l\omega - g_{lj}\partial^i\omega] - n\nabla_l\partial_j\omega = \\ &= \nabla_l\partial_j\omega + \nabla_j\partial_l\omega - g_{lj}\nabla_i\partial^i\omega - n\nabla_l\partial_j\omega\end{aligned}$$

and since

$$\nabla_l\partial_j\omega = \partial_l\partial_j\omega - \Gamma_{lj}^\alpha\partial_\alpha\omega = \nabla_j\partial_l\omega,$$

we have

$$\nabla_iS_{lj}^i - \nabla_lS_{ij}^i = -g_{lj}\nabla_\alpha\partial^\alpha\omega + (2-n)\nabla_l\nabla_j\omega.$$

Then

$$\begin{aligned}
S_{i\alpha}^i S_{lj}^\alpha - S_{l\alpha}^i S_{ij}^\alpha &= n\partial_\alpha\omega[\delta_l^\alpha\partial_j\omega + \delta_j^\alpha\partial_l\omega - g_{lj}\partial^\alpha\omega] - [\delta_l^i\partial_\alpha\omega + \delta_\alpha^i\partial_l\omega - g_{l\alpha}\partial^i\omega][\delta_i^\alpha\partial_j\omega + \delta_j^\alpha\partial_i\omega - g_{ij}\partial^\alpha\omega] = \\
&= 2n\partial_l\omega\partial_j\omega - ng_{lj}\partial_\alpha\omega\partial^\alpha\omega - \partial_l\omega\partial_j\omega - \partial_l\omega\partial_j\omega + g_{jl}\partial_\alpha\omega\partial^\alpha\omega - n\partial_j\omega\partial_l\omega - \partial_j\omega\partial_l\omega + \\
&\quad + \partial_l\omega\partial_j\omega + \partial_l\omega\partial_j\omega + g_{lj}\partial^i\omega\partial_i\omega - \partial_j\omega\partial_l\omega = (n-2)(\partial_l\omega\partial_j\omega - g_{jl}\partial_\alpha\omega\partial^\alpha\omega).
\end{aligned}$$

4. For the scalar curvature we obtain:

$$\begin{aligned}
e^{2\omega}\tilde{R} &= R - g^{jl}g_{jl}\nabla^\alpha\partial_\alpha\omega - (n-2)g^{jl}\nabla_l\partial_j\omega + (n-2)g^{jl}\partial_j\omega\partial_l\omega - (n-2)g^{il}g_{jl}\partial_\alpha\omega\partial^\alpha\omega = \\
&= R - n\nabla^\alpha\partial_\alpha\omega - (n-2)\partial^\alpha\partial_\alpha\omega + (n-2)\partial^\alpha\omega\partial_\alpha\omega - n(n-2)\partial^\alpha\omega\partial_\alpha\omega = \\
&= R - 2(n-1)\nabla_\alpha\partial_\alpha\omega - (n-1)(n-2)\partial^\alpha\omega\partial_\alpha\omega.
\end{aligned}$$

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