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Lecture notes

Conformal geometry and Riemann surfaces

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Lectures 9-10

Beltrami equation

Starting from this Lecture we will concentrate on the case $n = 2$.

The aim of the next two Lectures is to prove that all 2-dimensional conformal Riemannian manifold are **conformally flat**. Our proof will follow the book by Vekua [1]

More precisely:

Theorem 1. *Let $(M^2, g_{ij}(\vec{x}))$ be a 2-dimensional Riemannian manifold, \vec{x}_0 be a point of M^2 . Then there exist a neighborhood $U(\vec{x}_0)$ and a local coordinate system (u, v) at U such that in the coordinates (u, v) the metric tensor has the form*

$$g_{ij}(u, v) = e^{2\omega(u, v)} \delta_{ij}, \quad (1)$$

where $\omega(u, v)$ is a scalar real function.

Definition 1. *Coordinates u, v such that $g_{ij}(u, v) = e^{2\omega(u, v)} \delta_{ij}$ are called **isothermal**.*

How to construct them?

Definition 2. *Let M^2 be a smooth **oriented** 2-dimensional manifold equipped with a Riemannian metric*

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

The **quasicomplex structure** on M^2 is a linear operator J acting on the tangent bundle TM^2 such that

1. For any point $\vec{x}_0 \in M^2$ it maps the tangent space to this point $T_{\vec{x}_0}M^2$ onto itself.
2. The restriction of J to the space $T_{\vec{x}_0}M^2$ is isometry with respect to the Riemannian metric g_{jk} ; moreover, it is the 90 degrees counterclockwise rotation.
3. $J^2 = -1$.

Proposition 1. *Let (x^1, x^2) be a **positive** coordinate system on M^2 . Then the we have the following formula for J :*

$$J = \frac{1}{\sqrt{|G|}} \begin{pmatrix} -g_{12} & -g_{22} \\ g_{11} & g_{12} \end{pmatrix}.$$

Proof. Consider the tangent space to a fixed point \vec{x}_0 . Let:

$$J = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Since $Je_1 = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$, $Je_2 = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$, we can explicitly write the conditions on J :

$$[a_{11}a_{21}] \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0, \quad [a_{12}a_{22}] \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

Hence, $a_{11}g_{11} + a_{21}g_{12} = 0$ and $\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = C_1 \begin{pmatrix} -g_{12} \\ g_{11} \end{pmatrix}$. Since we want a rotation in a positive direction, $C_1 > 0$. Similarly, $\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = C_2 \begin{pmatrix} -g_{22} \\ g_{12} \end{pmatrix}$, $C_2 > 0$. Then we have

$$C_1^2[-g_{12} \ g_{11}] \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} -g_{12} \\ g_{11} \end{pmatrix} = g_{11}$$

Then

$$C_1^2[0 \ g_{11}g_{22} - g_{12}^2] \begin{pmatrix} -g_{12} \\ g_{11} \end{pmatrix} = g_{11}.$$

Therefore,

$$C_1 = \frac{1}{g_{11}|G|}.$$

Finally,

$$Je_1 = \frac{1}{\sqrt{|G|}} \begin{pmatrix} -g_{12} \\ g_{11} \end{pmatrix},$$

$$Je_2 = \frac{1}{\sqrt{|G|}} \begin{pmatrix} -g_{22} \\ g_{12} \end{pmatrix}.$$

Hence, we proved that

$$J = \frac{1}{\sqrt{|G|}} \begin{pmatrix} -g_{12} & -g_{22} \\ g_{11} & g_{12} \end{pmatrix}.$$

□

One can easily check that

$$J^2 = \frac{1}{|G|} \begin{pmatrix} g_{12}^2 - g_{11}g_{22} & 0 \\ 0 & g_{12}^2 - g_{11}g_{22} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now we ask ourselves: how to find the isothermic coordinates? If coordinates (u, v) are isothermal, then

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let us introduce complex coordinate: $w = u + iv$. We already know $\partial_{\bar{z}}z = 0$. One can write

$$\partial_{\bar{w}} = \frac{1}{2} [\partial_u + i\partial_v] = \frac{1}{2} [\partial_u + iJ\partial_u].$$

It is easy to check that

Lemma 1. *Let a vector field $\vec{W} \neq 0$ in a neighborhood of a point \vec{x}_0 . Then a local coordinate system (u, v) is isothermal iff the function $w = u + iv$ satisfies Beltrami equation*

$$[L_{\vec{W}} + L_{J\vec{W}}]w = 0, \quad (2)$$

where $L_{\vec{W}}$ denotes the directional derivative along the vector field \vec{W} .

Therefore, to construct local isothermal coordinates near a point \vec{x}_0 , it is sufficient to construct local solution of the Beltrami equation such that $L_{\vec{W}}w \neq 0$ at the point \vec{x}_0 and to define coordinates (u, v) by

$$u = \operatorname{Re} w, \quad v = \operatorname{Im} w.$$

Remark 1. *Here and thereafter during these two lectures we use the following agreement: to simplify notations we write $f(z)$ instead of $f(z, \bar{z})$ without assuming that $\partial_{\bar{z}}f(z) = 0$, unless it is stated explicitly. Equivalently, notations $f(z)$ **does not** imply that $f(z)$ is complex-analytic.*

Our next step is to prove existence of Beltrami equation solutions!

Beltrami equation

Let (x, y) be local coordinates near the point \vec{x}_0 . We can choose $\vec{W} = \partial_x$, and the Beltrami equation takes the form

$$\bar{\mathcal{D}}w = 0, \quad \text{where } \bar{\mathcal{D}} = \partial_x + iJ\partial_x. \quad (3)$$

Using Propositions 1 we obtain

$$\bar{\mathcal{D}} = \partial_x + \frac{i}{|G|}[-g_{12}\partial_x + g_{11}\partial_y].$$

Using that

$$\partial_x = \partial_z + \partial_{\bar{z}}, \quad \partial_y = i[\partial_z - \partial_{\bar{z}}]$$

we obtain

$$\bar{\mathcal{D}} = \partial_z + \partial_{\bar{z}} - \frac{ig_{12}}{\sqrt{|G|}}[\partial_z + \partial_{\bar{z}}] + \frac{i}{\sqrt{|G|}}ig_{11}[\partial_z - \partial_{\bar{z}}] = a(z)\partial_{\bar{z}} + b(z)\partial_z,$$

where

$$a(z) = \left[1 + \frac{g_{11}}{\sqrt{|G|}} - \frac{ig_{12}}{\sqrt{|G|}} \right], \quad b(z) = \left[1 - \frac{g_{11}}{\sqrt{|G|}} - \frac{ig_{12}}{\sqrt{|G|}} \right].$$

$g_{11} > 0$, therefore

$$|\operatorname{Re} a(z)| > |\operatorname{Re} b(z)|, \quad |\operatorname{Im} a(z)| = |\operatorname{Im} b(z)|,$$

and

$$|\alpha(z)| < 1, \quad \text{where } \alpha(z) = \frac{a(z)}{b(z)}.$$

$|a| > 0$, therefore Equation 3 is equivalent to

$$\frac{1}{a} \bar{\mathcal{D}} w = 0,$$

and finally we transform Beltrami equation to the following form

$$[\partial_{\bar{z}} + \alpha(z)\partial_z]w(z) = 0. \quad (4)$$

We are looking for non-zero solutions of this equation. It is convenient to replace this differential equation by an integral one. Let us try to find a solution of (4) in the following form:

$$w(z) = z + \partial_{\bar{z}}^{-1} f(z), \quad (5)$$

Equation 4 is equivalent to

$$f(z) = -\alpha(z) - \alpha(z) \partial_z \partial_{\bar{z}}^{-1} f(z). \quad (6)$$

It is natural to solve (7) using the standard iteration procedure

$$f_0(z) = -\alpha(z), \quad f_{k+1}(z) = -\alpha(z) - \alpha(z) \partial_z \partial_{\bar{z}}^{-1} f_k(z, \bar{z}). \quad (7)$$

To prove that the iterative procedure converges, it is necessary to show that the norm of integral operator is smaller than 1. It is important to choose a proper functional space.

The main functional spaces. Denote by $C_{1/2}(\mathbb{C})$ the following subspace in the space of continuous functions on \mathbb{C} :

1. All functions $f(z) \in C_{1/2}(\mathbb{C})$ are bounded:

$$\sup_{z \in \mathbb{C}} |f(z)| = \mathcal{C}(f) < \infty. \quad (8)$$

2. All functions $f(z) \in C_{1/2}(\mathbb{C})$ satisfy the Hölder condition with exponent 1/2:

$$\sup_{\substack{z_1, z_2 \in \mathbb{C}, \\ z_1 \neq z_2}} \frac{|f(z_1) - f(z_2)|}{\sqrt{|z_1 - z_2|}} = \mathcal{H}(f) < \infty. \quad (9)$$

The norm on $C_{1/2}(\mathbb{C})$ is defined by

$$\|f\|_{C_{1/2}} = \sup_{z \in \mathbb{C}} |f(z)| + \sup_{\substack{z_1, z_2 \in \mathbb{C}, \\ z_1 \neq z_2}} \frac{|f(z_1) - f(z_2)|}{\sqrt{|z_1 - z_2|}} = \mathcal{C}(f) + \mathcal{H}(f) < \infty. \quad (10)$$

It is easy to check that

Lemma 2. 1. $C_{1/2}(\mathbb{C})$ is a Banach space with respect to the norm $\|\dots\|_{C_{1/2}}$.

$$2. \|fg\|_{C_{1/2}} \leq \|f\|_{C_{1/2}} \|g\|_{C_{1/2}}.$$

Let D be a disk in the plane $|z| \leq 2\mathcal{R}$, $\mathcal{R} > 0$. Denote by $C_{1/2}^0(D)$ the subspace of $C_{1/2}(\mathbb{C})$ formed by functions with support on D :

$$f(z) \in C_{1/2}(\mathbb{C}) \text{ belongs to } C_{1/2}^0(D) \text{ iff } f(z) \equiv 0 \text{ for } z \notin D. \quad (11)$$

Consider the following operators from $C_{1/2}^0(D)$ to $C_{1/2}(\mathbb{C})$:

$$(\partial_{\bar{z}}^{-1}f)(z) = \frac{1}{\pi} \iint_{w \in D} \frac{f(w) du \wedge dv}{z - w}, \text{ where } w = u + iv, \quad (12)$$

$$(\Pi f)(z) = (\partial_z \partial_{\bar{z}}^{-1} f)(z) = -\frac{1}{\pi} \iint_{w \in D} \frac{f(w) du \wedge dv}{(z - w)^2}, \text{ where } w = u + iv, \quad (13)$$

Let us remark that in (12), (13) we can integrate over any subset of \mathbb{C} containing D .

In contrast with (12), the integral in (13) does not converge absolutely, and requires a regularization. We shall use the following one:

$$(\Pi f)(z) = -\frac{1}{\pi} \lim_{\varepsilon \rightarrow +0} \iint_{\substack{w \in D \\ |w-z| \geq \varepsilon}} \frac{f(w) du \wedge dv}{(z - w)^2}, \text{ where } w = u + iv, \quad (14)$$

Lemma 3. *The operator $\Pi : C_{1/2}^0(D) \rightarrow C_{1/2}(\mathbb{C})$ is well-defined and bounded. Of course, the norm of this operator depends on \mathcal{R} .*

Proof of Lemma 3.

Step 1. To check that the integral operator defined by (14) is well-defined, we use the following formulas. Let $\chi(z)$ be the characteristic function for the disk $|z| \leq R$:

$$\chi(z) = \begin{cases} 1 & |z| \leq R, \\ 0 & |z| > 2R. \end{cases} \quad (15)$$

Then

$$(\partial_{\bar{z}}^{-1}\chi)(z) = \begin{cases} \bar{z} & |z| \leq R, \\ \frac{R^2}{z} & |z| > R. \end{cases} \quad (16)$$

$$(\Pi\chi)(z) = \begin{cases} 0 & |z| \leq R, \\ -\frac{R^2}{z^2} & |z| > R. \end{cases} \quad (17)$$

Therefore for $z \in \mathcal{D}$ we can write:

$$(\Pi f)(z) = -\frac{1}{\pi} \iint_{|w| \leq 4\mathcal{R}} \frac{(f(w) - f(z)) du \wedge dv}{(z - w)^2}, \text{ where } w = u + iv. \quad (18)$$

Here we increased the integration area to avoid problems with the estimates on the boundary.

From (18) we immediately obtain: for $|z| \in D$

$$|(\Pi f)(z)| \leq \frac{1}{\pi} \iint_{|w| \leq 4\mathcal{R}} \frac{\mathcal{H}(f) \sqrt{|z-w|} du \wedge dv}{|(z-w)|^2}. \quad (19)$$

By introducing polar coordinates

$$w = z + re^{i\phi}, \quad du \wedge dv = r dr \wedge d\phi \quad (20)$$

we obtain

$$|(\Pi f)(z)| \leq \frac{1}{\pi} \int_0^{2\pi} d\phi \int_0^{6\mathcal{R}} \frac{\mathcal{H}(f) r \sqrt{r}}{|r^2|} dr = 2 \int_0^{6\mathcal{R}} \frac{\mathcal{H}(f)}{\sqrt{r}} dr = 4\sqrt{6\mathcal{R}} \mathcal{H}(f) \leq \mathcal{C}_1(R) \|f\|_{C_{1/2}}. \quad (21)$$

Step 2. Let $z_1, z_2 \in D$, $z_0 = (z_1 + z_2)/2$, $|z_2 - z_1| = \varepsilon < \mathcal{R}/10$.

$$(\Pi f)(z_2) - (\Pi f)(z_1) = -\frac{1}{\pi} \iint_{|w| \leq 4\mathcal{R}} \left[\frac{(f(w) - f(z_2))}{(z_2 - w)^2} - \frac{(f(w) - f(z_1))}{(z_1 - w)^2} \right] du \wedge dv = I_1 + I_2, \quad (22)$$

where

$$I_1 = \frac{1}{\pi} \iint_{|w-z_0| \leq 10\varepsilon} \left[\frac{(f(w) - f(z_1))}{(z_1 - w)^2} - \frac{(f(w) - f(z_2))}{(z_2 - w)^2} \right] du \wedge dv, \quad (23)$$

$$I_2 = \frac{1}{\pi} \iint_{\substack{|w| \leq 4\mathcal{R} \\ |w-z_0| > 10\varepsilon}} \left[\frac{(f(w) - f(z_1))}{(z_1 - w)^2} - \frac{(f(w) - f(z_2))}{(z_2 - w)^2} \right] du \wedge dv, \quad (24)$$

Using the same arguments as at Step 1, we obtain:

$$|I_1| \leq 4 \int_{|r| \leq 11\varepsilon} \frac{\mathcal{H}(f)}{\sqrt{r}} dr = 8\sqrt{11\varepsilon} \mathcal{H}(f) \leq \mathcal{C}_2(R) \sqrt{|z_2 - z_1|} \|f\|_{C_{1/2}} \quad (25)$$

$$\begin{aligned} I_2 &= \frac{1}{\pi} \iint_{\substack{|w| \leq 4\mathcal{R} \\ |w-z_0| > 10\varepsilon}} \left[\frac{(f(w) - f(z_0))}{(z_1 - w)^2} - \frac{(f(w) - f(z_0))}{(z_2 - w)^2} \right] du \wedge dv = \\ &= \frac{1}{\pi} \iint_{\substack{|w| \leq 4\mathcal{R} \\ |w-z_0| > 10\varepsilon}} (f(w) - f(z_0)) \left[\frac{1}{(z_1 - w)^2} - \frac{1}{(z_2 - w)^2} \right] du \wedge dv. \end{aligned} \quad (26)$$

Let us shift the integration variables:

$$\tilde{w} = w - z_0, \quad \tilde{z}_1 = z_1 - z_0, \quad \tilde{z}_2 = z_2 - z_0 = -\tilde{z}_1,$$

and assume that $\tilde{w} = \tilde{u} + i\tilde{v}$. Then

$$\begin{aligned} I_2 &= \frac{1}{\pi} \iint_{\substack{|\tilde{w}+z_0|\leq 4\mathcal{R} \\ |\tilde{w}|>10\varepsilon}} (f(\tilde{w}+z_0) - f(z_0)) \left[\frac{1}{(\tilde{w}-\tilde{z}_1)^2} - \frac{1}{(\tilde{w}+\tilde{z}_1)^2} \right] d\tilde{u} \wedge d\tilde{v} = \\ &= \frac{1}{\pi} \iint_{\substack{|\tilde{w}+z_0|\leq 4\mathcal{R} \\ |\tilde{w}|>10\varepsilon}} (f(\tilde{w}+z_0) - f(z_0)) \left[\frac{4\tilde{z}_1\tilde{w}}{(\tilde{w}-\tilde{z}_1)^2(\tilde{w}+\tilde{z}_1)^2} \right] d\tilde{u} \wedge d\tilde{v}, \end{aligned} \quad (27)$$

therefore

$$|I_2| \leq \frac{1}{\pi} \iint_{\substack{|\tilde{w}+z_0|\leq 4\mathcal{R} \\ |\tilde{w}|>10\varepsilon}} \mathcal{H}(f) \sqrt{|\tilde{w}|} \left[\frac{10}{9} \right]^4 \left[\frac{2\varepsilon|\tilde{w}|}{|\tilde{w}^4|} \right] d\tilde{u} \wedge d\tilde{v}, \quad (28)$$

Let

$$\tilde{w} = re^{i\phi}, \quad d\tilde{u} \wedge d\tilde{v} = r dr \wedge d\phi$$

Then

$$\begin{aligned} |I_2| &\leq \frac{1}{\pi} \int_0^{2\pi} d\phi \int_{10\varepsilon}^{6\mathcal{R}} \mathcal{H}(f) \sqrt{r} \left[\frac{10}{9} \right]^4 \left[\frac{2\varepsilon r}{r^4} \right] r dr = 4\varepsilon \left[\frac{10}{9} \right]^4 \mathcal{H}(f) \int_{10\varepsilon}^{6\mathcal{R}} r^{-3/2} dr \leq \\ &\leq 4\varepsilon \left[\frac{10}{9} \right]^4 \mathcal{H}(f) \frac{2}{\sqrt{10\varepsilon}} \leq \frac{4}{\sqrt{10}} \left[\frac{10}{9} \right]^4 \mathcal{H}(f) \sqrt{\varepsilon} \leq \mathcal{C}_3(R) \sqrt{|z_2 - z_1|} \|f\|_{C_{1/2}} \end{aligned} \quad (29)$$

Therefore

$$|(\Pi f)(z_2) - (\Pi f)(z_1)| \leq (\mathcal{C}_2(R) + \mathcal{C}_3(R)) \|f\|_{C_{1/2}} \sqrt{|z_2 - z_1|}. \quad (30)$$

This completes the proof.

End of proof of Lemma 3.

Let us return to constructing **local solutions** of Beltrami equation

$$[\partial_{\bar{z}} + \alpha(z)\partial_z]w(z) = 0. \quad (31)$$

To simplify notations, let us assume that we construct solutions in a small neighborhood of the point $z = 0$. Let us point out that:

1. Inside the disk $|z| \leq \mathcal{R}$ Equation (31) is equivalent to

$$[\partial_{\bar{z}} + \chi(z)\alpha(z)\partial_z]w(z) = 0. \quad (32)$$

where $\chi(z)$ is the same as in (15).

2. It is convenient to apply the following change of variables:

$$\tilde{z} = z - \alpha(0)\bar{z}, \quad \bar{\tilde{z}} = -\overline{\alpha(0)}z + \bar{z}. \quad (33)$$

In these new coordinates Equation (31) takes the form:

$$[\partial_{\tilde{z}} + \tilde{\alpha}(\tilde{z})\partial_{\tilde{z}}] w(\tilde{z}) = 0, \quad \text{where} \quad \tilde{\alpha}(\tilde{z}) = \frac{\alpha(z(\tilde{z})) - \alpha(0)}{1 - \alpha(z(\tilde{z}))\overline{\alpha(0)}}. \quad (34)$$

Let us remark that $\tilde{\alpha}(0) = 0$.

3. The dilation

$$\tilde{z} = \lambda \tilde{\tilde{z}}, \quad \lambda \in \mathbb{R}, \quad \lambda > 0, \quad (35)$$

results in the following transformation of the functions α in Beltrami equation:

$$\tilde{\tilde{\alpha}}(\tilde{\tilde{z}}) = \tilde{\alpha}(\lambda \tilde{\tilde{z}}). \quad (36)$$

It is easy to prove the following:

Lemma 4. 1. For any $\lambda \in \mathbb{R}, \lambda > 0$

$$\chi(\tilde{\tilde{z}})\tilde{\tilde{\alpha}}(\tilde{\tilde{z}}) \in C_{1/2}^0(D). \quad (37)$$

2. For any $\delta > 0$ there exists $\lambda \in \mathbb{R}, \lambda > 0$ such that

$$\|\chi(\tilde{\tilde{z}})\tilde{\tilde{\alpha}}(\tilde{\tilde{z}})\|_{C_{1/2}} < \delta. \quad (38)$$

We construct a solution of (31) inside the disk $|z| \leq \mathcal{R}$ in the following form

$$w(z) = z + \partial_{\bar{z}}^{-1} f(z), \quad (39)$$

where $f(z)$ satisfies

$$f(z) = -\alpha(z) - \chi(z)\alpha(z) \Pi \phi(z). \quad (40)$$

From Lemmas 3,4 it follows that using the coordinate changes (35), (36), the norm of the function $\alpha(z)$ and the norm of the operator $\chi(z)\alpha(z) \Pi \phi(z)$ can be made smaller than 1/4. Then the iterative procedure converges, $\|f(z)\| \in C_{1/2}^0(D)$ and $|f(z)| < 1/2$. Therefore the function $w(z)$ is continuously differentiable, satisfy Beltrami equation and

$$\det \begin{vmatrix} \frac{\partial w}{\partial z} & \frac{\partial w}{\partial \bar{z}} \\ \frac{\partial \bar{w}}{\partial z} & \frac{\partial \bar{w}}{\partial \bar{z}} \end{vmatrix} > 0,$$

therefore w defines a local isothermal (conformally flat) coordinate near 0. The transition functions are either holomorphic or antiholomorphic. If the manifold is orientable and all original coordinate systems are positive, the transition functions are holomorphic, and we obtain a complex structure from the conformal one.

We proved the following theorem:

Theorem 2. *In dimension $n = 2$ all Riemannian manifolds are conformally flat.*

References

- [1] Vekua, I.N., Generalized analytic functions. Oxford: Pergamon Press, 1962